The Roche Limit

For an object that is held together by its own gravity, there is a limit to the tidal gravity it can survive. This is often translated into a lower limit on the distance the object can be from a more massive source of gravity. As we will learn in this supplement, at a deeper level of insight the survival criterion can be presented as a comparison between the average *density* of the object and the average density you would have if you spread out the larger mass uniformly over the orbit of the object. Let's explain.

1. The Basics

In this course we have encountered *tidal force*. From section 3 of Supplement 8 we recall that the magnitude of the tidal force for an object of mass m and radius R a distance $r \gg R$ from a mass $M \gg m$ is

$$F_{\rm tide} = \frac{2RGMm}{r^3} \ . \tag{1}$$

If the object is held together entirely by its own gravity, then when the tidal force exceeds the self-gravity force of the object, the object will be ripped apart. The self-gravity force is

$$F_{\rm self} = \frac{Gm^2}{R^2} \ . \tag{2}$$

We will set the two equal to each other and solve for r. This will give us the radius $r_{\rm tide}$ such that if the orbital radius $r < r_{\rm tide}$, a self-gravitating object will be torn apart.

$$\begin{aligned} F_{\rm tide} &= F_{\rm self} \\ \tfrac{2RGMm}{r^3} &= \tfrac{Gm^2}{R^2} \\ \tfrac{2RM}{r^3} &= \tfrac{m}{R^2} \\ r^3 &= 2R^3(M/m) \\ r_{\rm tide} &= R(2M/m)^{1/3} \ . \end{aligned} \tag{3}$$

The radius $r_{\rm tide}$ is given various names, including the Roche limit, after the French astronomer Édouard Roche, who computed the limit in 1848.

2. Relation to density

Looking back at our derivation, we see that we can go another way:

$$\begin{aligned} \tfrac{2RM}{r^3} &= \tfrac{m}{R^2} \\ \tfrac{M}{r^3} &= \tfrac{1}{2}\tfrac{m}{R^3} \ . \\ \tfrac{M}{(4\pi/3)r^3} &= \tfrac{1}{2}\tfrac{m}{(4\pi/3)R^3} \\ \bar{\rho}_M(r) &= \tfrac{1}{2}\bar{\rho}_m \ . \end{aligned} \tag{4}$$

This says that our criterion can be rephrased. A self-gravitating object is torn apart by tidal forces when the average density of matter inside its orbit is about half the average density of the object. That is, if you were to spread the mass M evenly over the whole sphere inside the object's orbital radius r to get an average density $\bar{\rho}_M(r)$, if that density is at least half of the average density $\bar{\rho}_m$ of the object then the object is doomed.

This is actually a pretty deep statement. It means that it is the average *density* of an object, rather than its radius and mass separately, that determines how close it can get to a massive body. Whether a moon has a radius of 10 km or 1000 km, its closest possible separation depends only on its density. Saturn's giant rings, which are far more massive than all other rings in the Solar System put together, might have originated as a moon that got too close.

Challenge: Saturn's rings are made of lots of individual particles which are boulder-sized or less. How can the boulders survive, given that this is inside the Roche limit?

As always, feel free to talk with the tutors, the TAs, or me about the topics in this supplement!

Practice problems

1. How close could the Moon get to the Earth before it is tidally torn apart?

Answer: Our formula for the tidal radius is $r_{\rm tide} = R(2M/m)^{1/3}$, where m and R are respectively the mass and radius of the object, and M is the mass of the thing it's moving around. Using $m = 7.35 \times 10^{22}$ kg and $R = 1.74 \times 10^{6}$ m for the Moon, and $M = 5.97 \times 10^{24}$ kg for the Earth, we get $r_{\rm tide} = 9.48 \times 10^{6}$ m.

2. Based on our formula, how close could *you* get to the Earth before you are tidally torn apart? Do you have any comment on this result?

3. How close could a clone of the Sun get to a million solar mass black hole before it is torn apart? We actually see these happening: they are called *tidal disruption events*.

4. How close could a clone of the Sun get to a *billion* solar mass black hole before it is torn apart? Compare this with the event horizon radius of a black hole ($R_{\rm horizon} = 2GM/c^2$ for a black hole of mass M) and comment on the result.

5. We effectively did this derivation by considering two objects that are not moving. But such objects would just fall toward each other; more realistically, the objects will orbit around each other. Redo the calculation above, but assume that the object m is moving in a circle around the more massive object M. To perform this calculation, you will need to include in your calculation the centrifugal "force" $m\Omega^2 r$ (where Ω is the angular velocity), and its derivative, as a function of radius. **Hint:** make the assumption that the mass m is tidally locked to M, which means that Ω is fixed at the orbital angular velocity at the *center* of the mass m. In that case, if r_0 is the orbital radius of the center of m, $\Omega = (GM/r_0^3)^{1/2}$ independent of radius.