Hydrostatic Equilibrium and Exponential Atmospheres

One of the fundamental principles underlying the study of atmospheres is that atmospheres support themselves against gravity by *pressure gradients*. A "gradient" is a change in a quantity with location; in this case, the atmospheric pressure decreases with increasing altitude.

That's what we'll discuss in this set of notes, but before we get into details there are some perspectives that are useful:

1. You might not think much about it, but atmospheric pressure is pretty consistent over the surface of the Earth at a given altitude. For example, the lowest sea level pressure not in a tornado that was ever recorded was at 86% of standard atmospheric pressure (according to Wikipedia, this was during a typhoon in 1979). Indeed, any significant pressure gradients at a given altitude result in high winds (hurricanes, typhoons) that move to re-establish equilibrium.
2. In contrast, there is very significant variation in pressure with altitude. The pressure at the top of Mount Everest is about 1/3 what it is at sea level. This is maintained steadily, without the necessity of large motions of the atmosphere. Thus over a comparatively small distance (just about 9 km), the pressure changes by far more than over the thousands of kilometers of the Earth.
3. Pressure itself is not enough to produce significant forces. Consider, for example, that the atmosphere around you has a pressure that would exert a force of about 14 pounds on every square inch (or about a kilogram per square centimeter, although we're mixing units). Over your whole body, that's thousands of pounds of force. And yet you aren't crushed to death. Why? Because you have an equal pressure inside you, and the net forces basically cancel out.
4. But pressure *differences* do matter. You experience this if you are, for example, stuffed up with a cold and take a flight. The air pressure within the cabin is reduced somewhat to lower the stress on the airplane (but not to anywhere close to the outside air pressure at 30,000 feet; you'd die). If you can't equalize the pressure inside and outside of your ears, you experience a painful net force.
5. **It is a pressure gradient, not just pressure by itself, that exerts force.** I will insist strongly on this in homeworks and exams. **If you say that pressure balances gravity, you'll get points off.**

1. Derivation of the Equation of Hydrostatic Equilibrium

Imagine a thin, horizontal slab of the atmosphere. This slab has a density ρ, a thickness dr (it goes from height r to height $r + dr$; note that as usual dr is a differential, so we're thinking of a very small thickness), and an area A. Let the pressure be $P(r)$ at height r (and thus $P(r + dr)$ at height $r + dr$), and let the gravitational acceleration be g, which points in the direction of smaller r.

The volume of the slab is $V = Adr$ and its mass is therefore $m = \rho V = \rho Adr$. Thus the gravitational force on the slab is $F_g = mg = \rho gAdr$. For the atmosphere to avoid falling, there must be a net zero force on the slab. The opposing force is from the pressure *gradient*, i.e., if the pressure acting on the bottom face of the slab (which acts in the direction opposite gravity) is enough larger than the pressure acting on the top face of the slab (which acts in the same direction as gravity) then the net pressure force will balance gravity.

To compute the force from the pressure gradient we note that pressure is force per area, so the pressure force upward at r is $P(r)A$. Similarly, the pressure force *downward* is $P(r + dr)A$. Thus the *net* pressure force upward is

$$\begin{aligned} F_P = P(r)A - P(r+dr)A &= -[P(r+dr) - P(r)]A \\ &= -AdP \; . \end{aligned} \tag{1}$$

In equilibrium, this needs to be equal to the downward force of gravity:

$$\begin{aligned} -AdP &= \rho gAdr \\ dP/dr &= -\rho g \; . \end{aligned} \tag{2}$$

This is the *equation of hydrostatic equilibrium.*

We can now understand why a helium balloon rises in the air. It's not that the balloon has negative weight; its mass is positive, so if you let a helium balloon go on the Moon it would fall at the same rate as anything else. It's that the pressure gradient force on a helium balloon is the same as it would be on a balloon filled with air (**Do you understand why?**), but the gravitational force on a helium balloon is less than the force on an air-filled balloon of the same volume because the density of helium gas is less than the density of air. Thus there is *not* a balance, and hence the balloon rises upward.

2. Derivation of Pressure Profile: ideal gas, constant T, g

Here we apply the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g \tag{3}$$

to a specific case that is applicable to atmospheres. We will assume that the temperature is constant, and so is the gravitational acceleration. This simplifies the derivation; we will revisit those assumptions after we are done.

In order to solve the equation of hydrostatic equilibrium, we need to assume some relation between the pressure P and the density ρ. Such a relation is called an *equation of state*. A very common equation of state to assume is an *ideal gas*; there are different ways to write the relation, but we will choose

$$P = nkT \; , \tag{4}$$

where n is the number density of molecules (i.e., the number per unit volume), k is Boltzmann's constant, and T is the temperature. The number density n is related to the mass density ρ (mass per unit volume) by $n = \rho/m$, where m is the mass of the molecule. Thus

$$\begin{aligned} P &= (\rho/m)kT = (kT/m)\rho \\ \rho &= \tfrac{m}{kT}P \; . \end{aligned} \tag{5}$$

Then:

$$\frac{dP}{dr} = -\rho g = -(mP/kT)g = -(mg/kT)P \; . \tag{6}$$

Moving some factors we get

$$\frac{dP}{P} = -\frac{mg}{kT}dr \; . \tag{7}$$

Now we assume that g and T are both independent of r. If so, then doing an indefinite integral we get

$$\ln P = \text{constant} - \frac{mg}{kT}r \tag{8}$$

or, taking the exponent of both sides and setting the exponent of the constant to P_0,

$$P = P_0 e^{-r/(kT/mg)} \; . \tag{9}$$

Here kT/mg has units of length; we can call this H, the *scale height*, to finally get the equation in the main slides:

$$P = P_0 e^{-r/H} \; . \tag{10}$$

Note that here $P = P_0$ at a height $r = 0$ which is a *reference height*. We can pick anything as a reference height, and the equation will still work. For example, sea level might be convenient (in which case, for the Earth, P_0 is one "atmosphere" of pressure), but a kilometer above sea level would work just as well, as long as we are clear about the convention we use.

For Earth, let's say that $T = 290$ K, $m = 4.65 \times 10^{-26}$ kg (the mass of a nitrogen molecule), and $g = 9.8$ m s^{-2}. Then the scale height we would compute from this formula is $H = kT/mg = 8.8$ km, just about the height of Mount Everest. This is, in fact, correct; at the top of Mount Everest the atmospheric pressure is about $e^{-1} \approx 37\%$ of what it is at sea level. It also tells us that if we can use our atmospheric pressure formula to much greater heights, then the pressure dies off pretty quickly; for example, at 100 km altitude we would expect a pressure that is a factor $e^{-100/8.8} \approx 10^{-5}$ times what it is at sea level.

But now let's revisit our assumptions for the atmosphere:

1. Is the atmosphere an ideal gas? For the Earth, the typical distance between molecules in the air is much larger than the sizes of the molecules, so this is a pretty good approximation. This also applies to the atmosphere of Mars (which is much thinner than ours) and to the *top* of any atmosphere. But when the density gets large (such as at the base of Venus' atmosphere, or deep inside Jupiter or the Sun), this approximation is not good anymore. Thus, as always, you need to think about whether an approximation works for your particular problem.

2. Is the gravitational acceleration constant? We found above that the scale height for Earth's atmosphere is around 9 km. The radius of the Earth is about 6,400 km. The inverse square law of gravity tells us that the magnitude of the acceleration of gravity is GM/r^2, where M is the mass of the Earth and r is the distance from its center. Thus a fractional change in radius of $9/6,400 \approx 0.14\%$ makes only a very small difference to g. Again, however, if we are thinking about a wide range of radii (for example, the interior of Saturn or of a star), this might not be a good approximation.

3. Is the temperature constant? No! At least, it isn't exactly constant. If you've ever climbed a mountain you know that temperature decreases as you go up (*very* high up, as in $>$ 80 km, the temperature goes back up; weirdly, this is related to the difficulty molecules have in cooling when the density is low!). Also, there is certainly a lot of temperature variation over the Earth; the record high is 330 K (Death Valley), and the record low is 178 K (Antarctica, obviously). The poles are colder, the equator is hotter, it depends on whether it is day or night, and so on. So how can we justify using a constant temperature?

   One reason is that at a given time, latitude, and longitude, the temperature really doesn't vary enormously with altitude (which is what we care about for our derivation). The atmosphere really *does* expand and contract from day to night. In addition, there is a lot of mixing that goes on; perhaps you've heard of winds? This is also why we use a single scale height for all molecules; for example, from looking at the formula you might imagine that the scale height of hydrogen molecules would be 14 times larger than the scale height of nitrogen molecules, because hydrogen is 14 times lighter than nitrogen, but the gases are all constantly being mixed. Thus the use of a "global average" temperature isn't too bad! However, just as with the other assumptions, you need to consider carefully in any given situation whether you can make the assumption safely.

Practice problems

1. At $T = 290$ K and one Earth gravity ($g = 9.8$ m s^{-2}), calculate the scale height of hydrogen gas, H_2.

Answer: We look up the mass of H_2 and find $m = 2 \times 1.67 \times 10^{-27}$ kg, or $m = 3.34 \times 10^{-27}$ kg. The scale height is $H = kT/mg$, so putting in the values we get $H = 122$ km. Note that we could also have obtained this by looking in the notes to find that $H = 8.8$ km for the $m = 4.65 \times 10^{-26}$ kg nitrogen molecule; $122 = 8.8 \times (4.65 \times 10^{-26}/3.34 \times 10^{-27})$.

2. For the same conditions as the first problem ($T = 290$ K and $g = 9.8$ m s^{-2}), compute the scale height of oxygen gas, O_2.

3. For the same conditions as the first problem, compute the scale height for tungsten hexafluoride (WF_6), which is the densest known gas at standard temperature and pressure. **Safety note:** do *not* play with WF_6, which is a toxic, corrosive gas that will dissolve you as soon as look at you (it forms hydrofluoric acid on contact with humidity).

4. Going back to the original equation of hydrostatic equilibrium, $dP/dr = -\rho g$, you see that the pressure increases more rapidly when the density is higher. Suppose we put in water (which has a density about 800 times that of air) instead of air. How far would you have to go down to increase the pressure by one atmosphere? Have you ever dived that far? If so, how did you survive?