ASTR 688R
Problem Set 4
Due Thursday, April 27

1. Derive the Schönberg-Chandrasekhar limit on the fractional mass in the core of a star, using a uniform density model. To do this, assume that the core has a mass M_c and radius R_c, with a composition that has a mean molecular weight of μ_c. Assume that the star as a whole has a mass M and a radius R (the constant-density assumption of course means that $M_c/R_c^3 = M/R^3$) and that the envelope (everything outside the core) has a mean molecular weight $\mu_{\rm env}$. Assume also that the core and the envelope are both ideal gases. Now consider the interface between the core and the envelope:

(a) Use the virial theorem in the form $2K + \Omega = 3P_sV$, where V is the volume of the core and K is the thermal energy, to derive the surface pressure P_s at R_c, as a function of the temperature T_0 of the core (assumed isothermal).

(b) Calculate the maximum pressure $P_{0,\rm max}$ by maximizing P_s as a function of R_c, keeping the other quantities constant.

(c) In the constant-density model, calculate the central pressure P_e and central temperature T_e for a star of mass M, radius R, and mean molecular weight $\mu_{\rm env}$. Setting $T_e = T_0$ (i.e., no temperature jump across the boundary), write P_e as a factor times T_0^4/M^2.

(d) From the condition $P_e \leq P_{0,\rm max}$, calculate an upper bound on M_c/M. If the envelope is pure ionized hydrogen and the core is pure ionized helium, what is the upper limit on M_c/M? It should be somewhat larger than the value ≈ 0.1 found using more sophisticated models.

2. You're on a committee for the review of NASA Astrophysics Theory Program proposals. Sure enough, one of the proposals for which you're primary reviewer is by Dr. I. M. N. Sane. He thinks he has the perfect dark matter candidate. He writes that microlensing observations from projects such as MACHO, OGLE, and EROS seem to have detected a population of objects with masses typically in the $0.1\,M_\odot - 0.5\,M_\odot$ range that may constitute a fair fraction of the mass in our galactic halo. His idea is that this is a new class of objects: monopole stars. For our purposes, magnetic monopoles are pointlike particles of mass mc^2=1000 GeV that behave like degenerate fermions. Estimate the maximum mass of stars composed of monopoles, given the limitation that the radius of the star cannot be less than the Schwarzschild radius. Use this to evaluate Dr. Sane's idea.

3. (adapted from the new version of Hansen & Kawaler) In general relativity, for nonrotating spherical stars the equation of hydrostatic equilibrium is modified to become

the Tolman-Oppenheimer-Volkoff (TOV) equation

$$\frac{dP}{dr} = -\frac{G[\rho(r) + P(r)/c^2][M_r + 4\pi r^3 P(r)/c^2]}{r[r - 2GM_r/c^2]} \tag{1}$$

where $\rho(r)$ is now the mass-energy density (not just the rest mass density), but we still have $M_r = \int 4\pi r^2 \rho \, dr$. Let's consider a constant-density "star". A solution of the TOV equation is then

$$P(r) = \rho c^2 \left[\frac{(1 - 2GMr^2/R^3c^2)^{1/2} - (1 - 2GM/Rc^2)^{1/2}}{3(1 - 2GM/Rc^2)^{1/2} - (1 - 2GMr^2/R^3c^2)^{1/2}} \right] , \tag{2}$$

where M is the total mass-energy of the star and R is its radius.

(a) Show that in the limit $c \longrightarrow \infty$ this reduces to the Newtonian form for the constant-density solution, and that the central pressure in that limit is retrieved.

(b) Define $\alpha = 2GM/Rc^2$. Show that the GR solution does odd things when $\alpha \longrightarrow 8/9$.

(c) If $\alpha = 8/9$, what is R in km as a function of $M/M_\odot$?

(d) If $M = M_\odot$, what is the density in g cm^{-3} at $\alpha = 8/9$?

4. One of the two astrophysically important parameters for black holes is the dimensionless angular momentum, $j = a/M = J/M^2$, where J is the angular momentum. For a black hole, j can be anywhere between 0 and 1, but not outside those limits.

(a) Calculate j for the Earth, assuming it has constant density.

(b) Suppose that a pre-supernova iron core has $M = 1.4\,M_\odot$, a radius R=1000 km and constant density. What is the upper limit to the angular velocity the core can have if it could, in principle, collapse directly to a black hole without losing angular momentum?