Problem Set 1
Due Thursday, February 17

1. (Problem 2 in Chapter 1 of Hansen & Kawaler) A useful (albeit not terribly realistic) model for a homogeneous composition star may be obtained by assuming that the density is a linear function of radius. Thus assume that

$$\rho(r) = \rho_c \left[1 - r/R\right] \tag{1}$$

where ρ_c is the central density and R is the total radius where zero boundary conditions, $P(R) = T(R) = 0$, apply.

(a) Find an expression for the central density in terms of R and M. (You will have to use the mass equation.)

(b) Use the equation of hydrostatic equilibrium and zero boundary conditions to find pressure as a function of radius. Your answer will be of the form $P(r) = P_c \times$(polynomial in r/R). What is P_c in terms of M and R? (It should be proportional to GM^2/R^4.) Express P_c numerically with M and R in solar units.

(c) In this model, what is the central temperature, T_c? (Assume an ideal gas.) Compare the result to that obtained for the constant-density model. Why is the central pressure higher for the linear model whereas the central temperature is lower?

(d) Verify that the virial theorem is satisfied and write down an explicit expression for Ω (i.e., what is q of Eq. 1.7?).

2. (This problem is due to Ellen Zweibel and is in the in-preparation new edition of Hansen & Kawaler). It appears that some stars (besides the sun) are orbited by planets. Those extra-solar planets discovered thus far seem to have masses comparable to, or greater than, that of Jupiter and they orbit the parent star close in. [My comment: beware of selection effects in matters

like these!] Suppose one of those planets is captured by, and accreted onto, the parent star. The way we imagine this to take place is that the planet's orbit is circular and just grazes the star before accretion. Once accretion has taken place and the planet is completely assimilated into the star, we expect the combined body to have a radius different than the original star. To get a common nomenclature, let M be the mass of the original star, m be the mass of the planet, and assume $m \ll M$. The radius of the parent star is R and what we wish to find is how much that changes in terms of the small quantity m/M. Call the change ΔR.

(a) As the planet is accreted the total energy of the star increases by some amount ΔW. Let this amount be due solely to the orbital energy of the planet just as it is accreted (i.e., neglect any chemical or gravitational energy addition from the planet). What is ΔW in terms of M, m, and R (and G)?

(b) If the gravitational potential energy Ω takes the form $\Omega = -qGM^2/R$, then find $\Delta\Omega$ to first order in the small quantities m/M and $\Delta R/R$. (Note that part of the change in Ω is due to the added mass m and part due to ΔR.) Assume q remains constant.

(c) Use the virial theorem in the form $\Delta W = \frac{3\gamma-4}{3(\gamma-1)}\Delta\Omega$ to finally solve for $\Delta R/R$.

(d) If $\gamma = 5/3$ and $q = 3/2$, what would $\Delta R/R$ be for the sun if it swallowed Jupiter?

3. In a star such as a white dwarf or a neutron star, the components providing the pressure (electrons for white dwarfs, neutrons for neutron stars) are degenerate throughout most of the star. This means that available levels are occupied up to some energy E_F (Chapter 3 has details), and has the consequence that there is virtually no dependence of pressure on temperature ($\chi_T = 0$ in equation 1.64). For nonrelativistic degeneracy ($E_F \ll mc^2$, where m is either the electron or neutron mass depending on

which provides the pressure), the pressure is proportional to the density to the 5/3 power. Using this and the homology relations, calculate the mass-radius relation for nonrelativistic stars that are (a) not convective, and (b) convective. (Adapted from Hansen & Kawaler problem 3, chapter 1).

4. A colleague of yours, Dr. I. M. N. Sane, is worried about global cooling. He points out that mass-energy equivalence means that as the Sun radiates it loses gravitational mass, and that low-mass objects orbiting high-mass objects (e.g., the Earth orbiting the Sun) approximately conserve their angular momentum even as the high-mass object loses mass. This means that the Earth will move out gradually and become cooler. Assuming that the Sun radiates at its current luminosity for 10^{10} years, that the Sun does not lose mass by any other means, and that the Earth's mass is unchanged, allay Dr. Sane's fears by calculating the change in the blackbody temperature at the Earth's orbital radius over (a) 10^{10} yr, and (b) Dr. Sane's $\sim 10^2$ yr lifetime.

Extra Credit (4 pts). Take the second variation of the total energy W (compare the discussion in §1.2) and derive a condition for stability. Assume that the equation of state is given by $P = c_1\rho^{\Gamma_1}$.

(a) Write $\delta^2 W$ in its general integral form, assuming this equation of state. Remember that $V_\rho = 1/\rho = \frac{dV}{dM_r}$. At the end it will be most convenient to change variables from M_r to V, so you will also need to express factors like $(\delta r)^2$ in terms of δV and V, where V is the regular volume.

(b) If the adiabatic index Γ_1 is a constant throughout the star, then it is possible to show that the volume perturbations are proportional to the volume: $\delta V = kV$, where k is a constant throughout the star. Use this to determine a simplified condition for stability.