Go over mean molecular weight again. It's the average mass per particle, in units of m_p. **Ask class:** derive the MMW for a bunch of examples (restrict it to those who missed something on the homework, marked by a MMW near their total score. Different person must answer for each question.). Examples: fully ionized $^6_3$Li (4 particles, 6 nucleon masses, 3/2), half-ionized $^4_2$He (2 particles, 4 nucleon masses, 2), fully ionized $^{12}_6$C (7 particles, 12 nucleon masses, 12/7). Then ask for MMW of a collection of equal numbers of H^+, OH^-, and H_2O (no electrons). MMW of ^{16}O and ^{1}H in this case is 36 nucleon masses divided by 3 particles, or 12.

The Sun: A Great Big Heap a' Friendly Self-Luminescent Gas

Primary source here is Hansen & Kawaler, chapter 8, with some material from chapter 10.

The Sun, being so close, allows us a far more detailed opportunity to study it than most stars allow. This means we can learn a great deal about the Sun that isn't available otherwise, and hence detailed aspects of stellar structure and evolution. **Ask class:** what are some of the things we can observe only on the Sun? Examples include neutrino fluxes, helioseismology, sunspots (allowing observations of rotation). We will consider some of the information available via these paths.

Much of the evolution of the Sun follows the low-mass path we discussed earlier. As in HK, we can take a somewhat different look as well.

Let's start with a uniform density star supported by ideal gas pressure. The internal energy density is

$$E = \frac{3}{2}nkT = \frac{3}{2}\rho\frac{kT}{\mu m_p} \ . \tag{1}$$

Note that Hansen & Kawaler write this as $(3/2)\rho N_A kT/\mu$. In cgs units, this is the same thing: the number of hydrogen atoms in one gram is N_A. The internal energy $U = EV$, and since $\rho V = M$, $U = \frac{3}{2}MkT/(\mu m_p)$. The virial theorem implies that $U = -\Omega/2$, which for a uniform density star is $\frac{3}{10}GM^2/R$. Equating, we find

$$T \approx 4\times 10^6 \mu \left(\frac{M}{M_\odot}\right)^{2/3} \rho^{1/3}\ \mathrm{K} \ . \tag{2}$$

You may reasonably object that the constant density approximation surely doesn't hold for the Sun, and aren't we about ready to give up these silly approximations? Well, sort of, but one can say more generally (from dimensional arguments) that $T \propto \mu M^{2/3}\rho^{1/3}$ is the scaling one expects from a large variety of models. So we'll go with that for the moment.

Comment: the following approach is simply to get a general idea of how the Sun's luminosity changes with time. Therefore, in order to get a quick estimate, we are going to make a number of approximations. These include dropping all constant factors, and using ratios instead of derivatives. This will give the overall view of the evolution: does the Sun become brighter or dimmer with time, and by roughly how much? The constant factors could be dropped anyway, because part of the scaling solution strategy is to normalize at the end using observations (e.g., the current luminosity of the Sun).

Slipping further into the debauchery of scaling approximations, we can take the energy transfer equation

$$L = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr} \tag{3}$$

and get some kind of average over the star by "cancelling the d's" and writing $L \propto RT^4/\kappa\rho$. Assuming a Kramers opacity, $\kappa = \kappa_0 \rho T^{-3.5}$, and using the mass equation $R \propto (M/\rho)^{1/3}$, we find

$$L \propto \frac{M^{5.33}\rho^{0.117}\mu^{7.5}}{\kappa_0}\ . \tag{4}$$

We can use this to figure out the way that the luminosity changes with time as the composition changes. Having sinned thus far against absolute precision, we've paved the way for future sins. In particular, those three significant figures in the exponents are rather silly. First, the mass of the Sun hardly changes at all during its main sequence lifetime (the solar wind spits out $\dot{M} \approx 10^{-14}\, M_\odot\ \mathrm{yr}^{-1}$), so we can ignore that factor. What about other factors? **Ask class:** how much do they expect that the Sun's luminosity has changed over the past four billion years? Not too much, otherwise life couldn't have arisen (if the Earth were frozen or boiled). We also know in advance that the radius hasn't changed much, so neither did ρ, and with a small exponent we can drop that factor as well. How about κ_0? **Ask class:** in the majority of the Sun, do they expect κ_0 to depend significantly on the hydrogen and helium fractions, given that bound-free dominates? No, it's mainly metals. So we pretend that κ_0 is more or less constant with time as well and end up with

$$\frac{L(t)}{L(0)} = \left[\frac{\mu(t)}{\mu(0)}\right]^{7.5}\ . \tag{5}$$

We therefore need to figure out how μ changes with time. We know that $\mu(t) = 4/(3+5X(t))$, so it's a matter of how rapidly the hydrogen is consumed. **Ask class:** If there is a luminosity $L(t)$ generated by hydrogen burning, and a release of $Q = 6\times10^{18}$ erg g^{-1}, what is the change in $X(t)$ with time for a star of mass M? It's $dX/dt = -L/MQ$. We then find

$$\frac{d\mu(t)}{dt} = \frac{5}{4}\mu^2(t)\frac{L(t)}{MQ}\ . \tag{6}$$

The differential equation for $L(t)$ becomes

$$\frac{dL(t)}{dt} = \frac{75}{8}\frac{\mu(0)}{MQ}\frac{L^{32/15}(t)}{L^{2/15}(0)}\ , \tag{7}$$

with the solution

$$L(t) = L(0)\left[1 - \frac{85}{8}\frac{\mu(0)L(0)}{MQ}t\right]^{-15/17} , \tag{8}$$

or for the Sun, with $t_\odot = 4.5 \times 10^9$ yr and $\mu(0) \approx 0.6$,

$$\frac{L(t)}{L_\odot} \approx \frac{L(0)}{L_\odot}\left[1 - 0.3\frac{L(0)}{L_\odot}\frac{t}{t_\odot}\right]^{-15/17} . \tag{9}$$

Fitting $L(t_\odot) = L_\odot$, $L(0) \approx 0.8\, L_\odot$. Now, incidentally, given our other crimes against exact solution, I personally would have made the approximation $32/15 \approx 2$ and $2/15 \approx 0$ in the differential equation for $L(t)$, so the final solution would have a -1 instead of a -15/17, but that's personal preference. So, it appears that the Sun initially had a luminosity about 20-30% less than it does now. As Hansen & Kawaler say, this might say interesting things about the conditions when life first formed 3.5 billion years ago.

Rotation

Taken primarily from Kippenhahn & Wiegert.

So what about rotation? So far we've ignored it completely for simplicity. But we know that stars rotate. What happens then? One consequence is that the equation of hydrostatic equilibrium has to be modified to take centrifugal forces into account. In that case, we would have $\nabla P = -\rho g_{\rm eff}$, where $g_{\rm eff}$ includes both gravity and centrifugal terms. The centrifugal acceleration for angular velocity ω a distance s from the rotation axis is $\omega^2 s\mathbf{e}_s$, where $\mathbf{e}_s$ is a unit vector perpendicular to the rotation axis, pointing outwards. Things are especially simple if the centrifugal acceleration can be derived from a potential:

$$\omega^2 s\mathbf{e}_s = -\nabla V . \tag{10}$$

This is equivalent to saying that the angular velocity depends only on s, so that it is constant on cylinders. This is called conservative rotation.

We can then combine the gravitational and centrifugal potentials: $\Psi = \phi + V$ (where $V = -\int_0^s \omega^2 s\, ds$) and rewrite the equation of hydrostatic equilibrium as

$$\nabla P = -\rho\nabla\Psi . \tag{11}$$

This means that the pressure gradient and potential gradient surfaces are parallel. Therefore, equipotential surfaces Ψ=constant correspond to surfaces of constant pressure, so pressure is a function of Ψ only. If we assume an ideal gas, in which $P = \rho/\mu m_p kT$, then T is also only a function of Ψ if the star is chemically homogeneous. Therefore, it too is constant on equipotential surfaces. If the star is not chemically homogeneous, it is T/μ that is constant on equipotentials. We can also take the curl of the equation of hydrostatic equilibrium to find $0 = -\nabla\psi \times \nabla\rho$, showing that the density is also constant on equipotential surfaces. We can also see that if the pressure is a function only of the density,

$P(\rho)$, then the rotation must be conservative, because $\nabla P(\rho)/\rho$ can be expressed as the gradient of a potential:

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{1}{\rho}\frac{dP}{d\rho}\frac{d\rho}{dr} = -\frac{d\Psi}{d\rho}\frac{d\rho}{dr} \ , \tag{12}$$

so $\Psi = -\int \frac{dP}{d\rho}(1/\rho)d\rho$. **Ask class:** what does this say about the rotation of white dwarfs? It should be conservative, and hence equal along cylinders, because the pressure is degeneracy pressure and therefore depends on density alone (to a good approximation).

Now let's consider what happens when we include radiative energy transport. The equation of transport can be written

$$\mathbf{F} = -\frac{c}{3\kappa\rho}\nabla(aT^4) = -\frac{c}{3\kappa\rho}\frac{d(aT^4)}{d\Psi}\nabla\Psi = -k(\Psi)\nabla\Psi \ , \tag{13}$$

because $\kappa = \kappa(\rho, T) = \kappa(\Psi)$. Here $-\nabla\Psi$ is the effective gravity including centrifugal terms. Now combine this with the equation of energy conservation, which we write in the form $\nabla\cdot\mathbf{F} = \epsilon\rho$:

$$\nabla\cdot\mathbf{F} = -\frac{dk}{d\Psi}\left(\nabla\Psi\right)^2 - k(\Psi)\nabla^2\Psi = -\frac{dk}{d\Psi}\left(\nabla\Psi\right)^2 - k(\Psi)\left(4\pi G\rho - \frac{1}{s}\frac{d(s^2\omega^2)}{ds}\right) = \epsilon\rho \ . \tag{14}$$

Here we've made use of Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$. Can this equation hold everywhere along equipotentials? No! Everything except the first term and the second term in parentheses is a function of Ψ only (remember $\epsilon = \epsilon(\rho, T) = \epsilon(\Psi)$), and is therefore constant along equipotentials. Consider now as an example something that rotates uniformly; then $(1/s)d(s^2\omega^2)/ds$ is constant, period. However, the first term, $-dk/d\Psi(g_{\rm eff})^2$, is definitely *not* constant; the effective gravity is greater at the poles than at the equator, along equipotentials.

This result, that when rotation enters that radiative transport and simple energy conservation can't be conserved simultaneously, is known as von Zeipel's paradox. The way out comes from the realization that we have overconstrained the problem; when rotation exists, another form of energy transport arises. This is meridional circulation. Note, by the way, that the nonconservation of energy under our strict assumptions goes like ω^2, as should be expected physically (a dependence on rotation to first order would imply that the *direction* of rotation makes a fundamental difference, which it can't). Meridional circulation also happens for non-conservative rotation. In a radiatively dominated region of the star (specifically where $(\nabla_{\rm ad} - \nabla)/\nabla_{\rm ad} \sim 1$), and assuming that the density in the region of interest is close to the mean density of the star, then the circulation time scale is approximately the *Eddington-Sweet* time scale

$$\tau_{\rm circ} \approx \frac{GM^2}{LR}\frac{1}{\chi} \approx \frac{\tau_{\rm KH}}{\chi} \ , \tag{15}$$

where $\chi \equiv \frac{\omega^2}{2\pi G\rho_c} \approx (\omega/\omega_K)^2$ is a parameter describing the importance of rotation. For the Sun $\chi \approx 10^{-5}$, and since $\tau_{\rm KH} \approx 10^7$ yr, circulation is not important in the deep envelope. However, near the surface, the time scale is shortened by a factor $\approx \rho/\bar{\rho}$, which can be tiny in the outer envelope. Meridional circulation is therefore important there.