Final exam: class vote means that it will be closed-book, like the midterm (constants and complicated equations will be provided; the four fundamental equations of stellar structure need to be remembered by the students).

Stellar Modeling

Primary source here is Hansen & Kawaler, chapter 7.

In principle, to construct a full stellar model we need to specify the total mass and the composition as a function of a coordinate such as radius (or total mass). Note also that we are still restricting ourselves to spherical, nonrotating models. We'll talk in a later lecture about some of the complexity introduced by rotation. We therefore need to know the microphysical relations

$$P(\rho, T, \mathbf{X}),\ E(\rho, T, \mathbf{X}),\ \kappa(\rho, T, \mathbf{X}),\ \epsilon(\rho, T, \mathbf{X})\ . \tag{1}$$

We also need some derivatives of these quantities. The relations to be satisfied are the fundamental equations of stellar physics. The structural equations are

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho\,, \quad \frac{dM_r}{dr} = 4\pi r^2\rho\,, \quad \frac{dL_r}{dr} = 4\pi r^2\epsilon\rho \tag{2}$$

or their Lagrangian equivalents. In addition, we need to know the run of temperature with pressure, or

$$\nabla \equiv \frac{d\ln T}{d\ln P} = -\frac{r^2 P}{GM_r\rho}\frac{1}{T}\frac{dT}{dr}\ . \tag{3}$$

If the run of temperature assuming radiative transfer

$$\nabla_{\rm rad} = \frac{3}{16\pi ac}\frac{P\kappa}{T^4}\frac{L_r}{GM_r} \tag{4}$$

is less steep than the adiabatic value, $\nabla_{\rm rad} < \nabla_{\rm ad}$, then $\nabla = \nabla_{\rm rad}$. If instead $\nabla_{\rm rad} > \nabla_{\rm ad}$, then $\nabla = \nabla_{\rm ad}$.

Right then, so we'd like to solve for the structure. But does a solution always exist? No. **Ask class:** can they think of a mass and composition that does not have an equilibrium solution? For example, there is no equilibrium $10\,M_\odot$ star composed entirely of iron (it would collapse, being in excess of the Chandrasekhar limit). If there is a solution, is it unique? No again. The reason turns out to be that the opacity and equation of state of matter is complicated enough that multiple solutions are possible. However, in practice the solution is unique, in the sense that only one of the possibilities is realistic. So we'll assume "practical uniqueness".

Polytropes

Before embarking on full stellar modeling, let's use a simplification and see how far that gets us. Let's assume that the equation of state is polytropic, meaning that the pressure is a power-law function of the density:

$$P(r) = K\rho^{1+1/n}(r)\ . \tag{5}$$

We talked about this approach in an earlier lecture. When it is put into the equation of hydrostatic equilibrium and dimensionless parameters are introduced, we end up with the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n\ . \tag{6}$$

Here the dimensionless coordinates θ and ξ are defined by $\rho(r) = \rho_c\theta^n(r)$ (ρ_c is the central density) and $r = r_n\xi$, where r_n is the scale length, $r_n^2 = (n+1)P_c/4\pi G\rho_c^2$. The boundary conditions reflect the physical constraints. If ρ_c is the central density, then $\theta(\xi = 0) = 1$. Spherical symmetry means that $dP/dr = 0$ at $r = 0$, so $\theta' = d\theta/d\xi = 0$ at $\xi = 0$. Finally, if we define the surface as where the pressure vanishes, then the surface must be at the first zero of $\theta(\xi)$. Calling this ξ_1, the last condition is $\theta(\xi_1) = 0$. Therefore, the total dimensional radius of the surface is

$$R = r_n\xi_1 = \left[\frac{(n+1)P_c}{4\pi G\rho_c^2}\right]^{1/2}\xi_1\ . \tag{7}$$

Analytic solutions are possible for n=0, 1, and 5, and are given in the book. Our primary interest is in $n = 3$ (a 4/3 power law) and $n = 3/2$ (a 5/3 power law). We therefore need appeal to numerical solutions.

Numerical solutions

WARNING! WARNING! DANGER WILL ROBINSON! We are now about to discuss how to solve the Lane-Emden equation numerically. The techniques discussed in HK are generally useful, but remember that the point is to solve the equations quickly and accurately. If this can be done by a simple, easily-coded algorithm, great! In particular, you can solve the LE equation easily by the crude technique of solving for $d^2\theta/d\xi^2$, then updating $d\theta/d\xi$ and θ as a function of ξ. It works. However, some of the other techniques are generally useful and can be applied in more complicated circumstances. I know that there is a strong tendency to simply import your favorite Numerical Recipes subroutine and pound away at an equation like this, but regardless of how you do it the most important thing is that it WORKS CORRECTLY! The way you test this is to look at limits, to compare with analytic solutions, and so on. Here, in the realm of numerics, is a place where you can easily go slack-jawed and drooling as you gaze vacantly at the screen of scrolling numbers, without understanding what to expect (I did this for $\sim$ 2 years while working on my thesis before finally getting it together!). So, with that warning, here we go...

Shooting methods.—These methods (generally, but also in this case) involve starting at one end of a domain where the solution is known, then integrating to the other end where the solution is not known. The simple integrator I mentioned above does this, and it is the basis of Runge-Kutta integrators as well. With RK integration, you have as an independent variable ξ and as dependent variables θ, θ', and θ''. Then, you integrate away with some step size h and weightings as determined in the RK method. The book describes the fourth-order method, but others are available.

A comment on this. RK integrators are fast, stable ways to solve many differential equations. However, whether you want to use them depends on the circumstance. Just as you wouldn't use an adaptive Reimannian routing to integrate a simple, smooth function (where a trapezoidal rule or even a rectangular integrator will do just as well and be easier to code), you shouldn't necessarily stampede straight for a RK. The reason is the the complexity of coding, although not much, introduces a small but finite error probability, which you have to weigh against the gains.

A different comment is that in the LE problem as well as some others you do have to be careful about where you star. If your first step is actually at $\xi = 0$, the equations are singular. You could just start at some small nonzero ξ, or you could use an analytic series expansion near $\xi = 0$:

$$\theta_n(\xi) = 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 - \ldots . \tag{8}$$

This type of series expansion is helpful in other circumstances as well. You can think of it as a way of getting the numerical solution off to a good start.

A last comment about shooting methods is that if the equations to be solved are somewhat unstable, shooting methods can err grievously. A place to be careful about that in the LE case is at the surface of the star, where the mass and density are small and larger errors are the result. It's not that bad, really, but some stiff differential equations are not amenable to shooting methods for this reason. Numerically, one way to tell if your shooting methods are good or bad is to try very slightly different initial conditions, or to change your resolution somewhat. If the solution at some point is changed drastically, you've got problems.

Fitting methods.—One way around this potential problem is to start a solution at both ends of the region, then try to match them in the center. This is called a fitting method, and it has similarities to root-finding using the Newton-Raphson method. Recall that in that circumstance, you guess at the location of the root, then correct using the value of the function and its first derivative. Here, suppose we define $x \equiv \xi$, $y \equiv \theta$, and $z \equiv d\theta/d\xi$. Integrate outward from $x = 0$ and inward from some $x = x_s$, which is your first guess as to where the surface is (therefore $y(x_s) = 0$, and you also need to compute z_s). Compare the two solutions at some fitting radius x_f. Barring supernaturally good intuition, the solutions won't match at x_f. Define $Y(x_s, z_s) = y_i(x_f) - y_o(x_f)$ and $Z(x_s, z_s) = z_i(x_f) - z_o(x_f)$. Continuity will eventually demand that $Y = Z = 0$. Now see what happens when x_s is varied by δx_s (keeping z_s constant) and when z_s is varied by δz_s (keeping x_s constant). This will produce changes in both Y and Z. You can therefore calculate numerical partial derivatives: $\partial Y/\partial x_s$, for example. You then expand a first-order Taylor expansion for $Y(x_s + \Delta x_s, z_s + \Delta z_s)$ (similar for Z). Since you want $Y = Z = 0$, you set this $Y(x_s + \Delta x_s, z_s + \Delta z_s)$ to zero, getting finally the two linear equations for Δx_s and Δy_s

$$\left(\frac{\partial Y}{\partial x_s}\right) \Delta x_s + \left(\frac{\partial Y}{\partial z_s}\right) \Delta z_s = -Y(x_s, z_s) \tag{9}$$

and

$$\left(\frac{\partial Z}{\partial x_s}\right) \Delta x_s + \left(\frac{\partial Z}{\partial z_s}\right) \Delta z_s = -Z(x_s, z_s) \tag{10}$$

Solving for $x_s + \Delta x_s$ and $z_s + \Delta z_s$, you iterate to a solution. The same principle can be applied to the real equations, in which (since it's a fourth-order differential system), four quantities must be specified (e.g., P and ρ at both the center and the surface). In this case, again you want to ensure continuity of quantities that must be continuous. This includes the pressure (otherwise infinite acceleration) and mass, but not density, because having, e.g., air on lead is fine!

Relaxation ("Henyey") schemes.—Another approach, in a sense a logical extension of the previous one, is a relaxation scheme. As a particular example, suppose you have the second-order system

$$\frac{dy}{dx} = f(x, y, z)\,, \qquad \frac{dz}{dx} = g(x, y, z) \tag{11}$$

with boundary conditions on y and z specified at the endpoints of an interval $x_1 \le x \le x_N$:

$$b_1(x_1, y_1, z_1) = 0\,, \qquad b_N(x_N, y_N, z_N) = 0\,. \tag{12}$$

In the usual case we assume all the x_i are specified (this is different from the Lane-Emden case where ξ_1 had to be determined, but we'll get to that in a second). The differential equations can be replaced by difference equations

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \frac{1}{2}(f_{i+1} + f_i)\,, \qquad \frac{z_{i+1} - z_i}{x_{i+1} - x_i} = \frac{1}{2}(g_{i+1} + g_i)\,. \tag{13}$$

We then make guesses at the correct y_i and z_i. These won't satisfy the equations, so we make corrections $y_i \longrightarrow y_i + \Delta y_i$, $z_i \longrightarrow z_i + \Delta z_i$, plug these into the equations again, and expand to first order (see book for the result). We then end up with equations for each of the i grid points, and can put these in the form of a matrix equation

$$\mathbf{M} \cdot \mathbf{U} = \mathbf{R}\,, \tag{14}$$

where $\mathbf{U}$ has the desired quantities Δy_i, Δz_i, $\mathbf{R}$ contains the boundary conditions, and $\mathbf{M}$ is the coefficient matrix that comes from linearizing the difference equations. Details about this are in the book. As with the fitting methods, after one round the y_i and z_i are corrected, hopefully we're closer to the answer than before, and the scheme is iterated. This can be a fast and rigorous way to solve equations, but nothing is guaranteed: in particular, if the initial guess is far enough off, the subsequent corrections may land the solution into some local minimum instead of the true solution.

If you don't know one end of the grid (as in the Lane-Emden problem), a way to deal with this is to define a variable $x = \xi/\xi_1 = \xi/\lambda$, so that the grid now goes from $x = 0$ to $x = 1$. This introduces another variable (λ), but also increases the number of boundary conditions (now $\theta = 0$ at $x = 1$ in addition to $y = 1,\ z = 0$ at $x = 0$), so the system of equations is closed.