**Black Holes**

We're now ready to start tackling the properties of black holes. As before, we'll start with a qualitative summary of their properties, then go into more quantitative detail.

(1) There is a unique spacetime outside spherically symmetric objects.

(2) This spacetime, conveniently represented with the Schwarzschild metric, has a "pit in the potential" compared to Newtonian gravity. That means that, in some sense, gravity is stronger than Newtonain gravity. The difference is negligible far out, but is important closer in.

(3) One of those differences is the presence of an event horizon. Nothing that passes across the event horizon can escape or send anything back to us. Coincidentally, this is at a radius $R = 2GM/c^2$ such that the Newtonian escape velocity is $c$.

(4) Another difference is that there are unstable circular orbits close enough to a black hole or neutron star. Far away, a slight perturbation of an orbit just makes it a little elliptical. Close enough, however, a slight perturbation causes it to spiral in quickly.

(5) Photons can be deflected dramatically close to a black hole. They can be bent by $2\pi$ or more. This causes strange optical effects that, unfortunately, aren't likely to be observed.

(6) Black holes are the simplest macroscopic objects in the universe. Their properties can be described by just their mass, their angular momentum, and their charge, and the charge has no important gravitational effect because it is quickly neutralized.

(7) If a black hole rotates, it drags spacetime with it (frame-dragging). This is true of any rotating object (cf Earth and GPB), but around a black hole the effects can be dramatic.

(8) There are laws of black holes comparable to the laws of thermodynamics. For example, if two black holes merge then, in analogy to the second law of thermodynamics, the total area of the final single hole is at least as great as the total area of the two initial holes combined.

(9) This thermodynamic analogy extends to black holes actually radiating (Hawking radiation), but the radiation is so slow it is negligible for any stellar-mass black hole (brane theory).

So now, let's think about the spacetime around black holes, and more generally, around anything spherically symmetric.

If the spacetime is spherically symmetric, then it is unique. This is an extension of the Newtonian result that the gravitational field outside a spherically symmetric object (even one moving, as long as the movement is spherically symmetric) depends only on the mass of the object. Birkhoff's theorem: the only vacuum, spherically symmetric grav. field is static. One convenient metric for this spacetime is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega \tag{1}$$

where $d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2$. Note here an important distinction: a *metric* is a description of a spacetime based on a particular choice of coordinates. The spacetime itself is more general. That is, the Minkowski metric describes spacetime in one way, but a coordinate transformation makes it look different even though the underlying physics is the same. That's important here, too. The Schwarzschild coordinates are convenient because they are chosen to represent important physical quantities: $t$ is the time between events measured by someone at infinity, $\theta$ and $\phi$ are their usual spherical coordinate equivalents, and $r$ is $1/2\pi$ times the circumference of a circle drawn at Schwarzschild coordinate at $r$. *However*, you have to be careful: the proper radial distance from $r_1$ to $r_2$ is *not* $r_2 - r_1$. It is $\int_{r_1}^{r_2} (g_{rr})^{1/2}\, dr$, which is larger in general. Looking at the Schwarzschild coordinates you have the impression that great badness happens at $r = 2M$, because the $dr^2$ term diverges there. In fact, that's the radius of the horizon, but nothing really becomes infinite.

Let us now relate the proper time $\tau$ of an observer at some radius $r$ in this spacetime to the proper time of an observer at infinity (which is the same as the Schwarzschild coordinate time $t$). For an observer fixed at $r, \theta, \phi$, $d\tau^2 = -ds^2 = (1 - 2M/r)\, dt^2$, so

$$d\tau = \left(1 - \frac{2M}{r}\right)^{1/2} dt\ . \tag{2}$$

This shows time dilation, as we expect it (distant observers see close observers moving slowly; close observers see distant observers moving fast). But there's an breakdown at $r = 2M$. This is the *event horizon*, which defines the "surface" of the black hole and defines the Schwarzschild radius. This is also called the *static limit*: static (non-moving) observers can't exist inside this radius. Now, this also means that if we at infinity watch something fall in to the black hole, it never appears to cross the horizon because of time dilation. It seems to "freeze" there, although it dims rapidly so we shouldn't expect to see a collection of frozen surprised aliens at the horizons of black holes! A person falling through $r = 2M$ does so in finite proper time, and does not feel infinite tidal forces (those happen at $r = 0$, which is a genuine singularity). Note that as $r \longrightarrow 2M$ the redshift approaches infinity.

The motion of test particles (i.e., those that react to but do not affect the spacetime) depends on whether the particles have zero or nonzero rest mass. Rather than deriving everything, let me just give a few of the important results. These are all for the Schwarzschild spacetime, i.e., without charge or rotation.

For a massive particle:

(1) A local static observer at $r$ measures the velocity of a radially freely falling particle released from rest at infinity as $v^{\hat{r}} = \sqrt{2M/r}$, which is exactly the Newtonian expression!

(2) An observer at infinity measures the orbital frequency of a particle in a circular orbit at Schwarzschild radius $r$ to be $\Omega = \sqrt{GM/r^3}$, which again is exactly the Newtonian expression!

(3) A particle in a circular orbit of radius $r$ has a specific angular momentum of

$$\ell^2 = \frac{Mr^2}{r - 3M} \ . \tag{3}$$

(4) A particle orbiting in a circle of radius $r$ has a specific energy relative to infinity of

$$e = \frac{r - 2M}{\sqrt{r(r - 3M)}} \ . \tag{4}$$

The specific energy is the energy per rest mass, so a particle at rest at infinity has $e = 1$.

(5) Particles in circular orbits with radii $r < 6M$ are unstable. Slight perturbations of such orbits cause them to spiral in rapidly towards the black hole (or neutron star, which can also be compact enough to allow unstable orbits). At this radius of the innermost stable circular orbit (ISCO), $\ell = \sqrt{12}M$ and $e = \sqrt{8/9}$. Therefore, if gas has spiraled in from infinity in nearly circular orbits, it has released $1 - \sqrt{8/9} \approx 0.057$ of its energy by the time it gets to the ISCO; inside the ISCO it probably releases little energy. Therefore, the efficiency of energy production is $\sim 10$ times the efficience of fusing hydrogen to helium! At the ISCO, a local static observer measures the orbital velocity as $v = c/2$, regardless of the mass of the black hole.

For a massless particle (a photon, to be definite):

(1) If a photon coming from infinity has an impact parameter less than $3\sqrt{3}M$, it will be captured by the black hole. The effective cross section of the black hole for photons is therefore $27M^2$.

(2) Closed orbits are of measure 0: a photon in a circular orbit with a radius $r = 3M$ will continue that way, but it's unstable. A perturbation out means that it escapes, a perturbation in means that it is captured.

The spacetime can also be written exactly, in closed form, if there is rotation. Metrics for this Kerr spacetime (e.g., the Boyer-Lindquist metric) are more complicated than is worth writing here, but it is useful to summarize. The single extra parameter for spinning black holes is a rotational parameter, $a \equiv J/M$. This parameter has the units of mass, and sometimes people use $j \equiv a/M$ for the dimensionless equivalent. When I mention things like the radius $r$ and the time $t$, I will use the Boyer-Lindquist values, which reduce to

Schwarzschild when $a = 0$ and have similar physical meaning (i.e., $t$ is the time measured by an observer at infinity, and $\phi$ has the usual meaning of an azimuthal coordinate).

The rotation means that spacetime is no longer spherically symmetric, although it is axisymmetric. The axisymmetry and stationarity (no time dependence) means that the angular momentum and energy are still conserved. The horizon moves into

$$r_+ = M + (M^2 - a^2)^{1/2} \ . \tag{5}$$

This means that $a < M$ for a black hole to exist (that is, for it to have an event horizon). If somehow it could have $a > M$, there would be no horizon, and would instead be a "naked singularity". A problem that is still unsolved is whether such naked singularities can exist. Penrose's "cosmic censorship conjecture" states that naked singularities are always clothed by horizons, but nothing has been proven.

A new, important radius that crops up is the radius of the "ergosphere". This is at

$$r_0 = M + (M^2 - a^2 \cos^2 \theta)^{1/2} \ . \tag{6}$$

Note that there is room between $r_0$ and the horizon radius $r_+$ for $a > 0$. Inside this radius, test particles (massive or massless) must rotate with the hole, even if they are outside the horizon. The ergosphere is therefore the static limit; things inside can't remain stationary with respect to infinity. This is an extreme example of frame-dragging. Even outside the ergosphere, spacetime is dragged in the direction of rotation of the hole. The natural frequency for this frame-dragging (i.e., the frequency of a particle with no angular momentum) is $\omega \approx \frac{2Ma}{r^3}$. The ergosphere is named thus ("ergo" being Greek for energy) because in principle something sent in there could split in two and emerge with more than the original energy of the particle, thereby tapping the spin energy of the hole. More astrophysically plausible is that threaded magnetic fields down the hole might extract its spin energy ("Blandford-Znajek process").

In a Kerr spacetime, the location of the ISCO is changed, as is the effeciency of energy extraction. For particles orbiting in the same direction as the hole is spinning, the ISCO moves in; for retrograde particles, the ISCO moves out. For prograde, energy extraction is more efficient if the hole is spinning faster, and the maximum is about 42%(!).

Hawking proved that in any interaction, the surface area of a black hole can never decrease. If many black holes are involved, it's the sum of the areas that can't decrease. This is analogous to the increase of entropy. The area of a Kerr black hole is

$$A = 8\pi M \left[ M + (M^2 - a^2)^{1/2} \right] \ , \tag{7}$$

which reduces to the expected $4\pi(2M)^2$ for $a = 0$. This means, in principle, that two equal-mass $a = M$ black holes with opposite spin could release 50% of their mass-energy in a collision, and two Schwarzschild black holes could release 29%. The actual fraction that would be released is a lot smaller, but not precisely known at this time (this should be one product of numerical simulations of merging black holes, but that's a tough enterprise).

Bekenstein tried to extend this thermodynamic analogy, but Hawking got the big prize: black holes have an effective temperature, and radiate! The temperature is

$$T = \frac{\hbar}{8\pi k M} = 10^{-7}\mathrm{K}\left(\frac{M_\odot}{M}\right) \ . \tag{8}$$

This is essentially because virtual particles can be separated by the gravitational field and one can be made real (analogous to pair creation in an electric field). It is therefore a remarkable marriage of quantum mechanics and general relativity. However, the timescale of emission is ridiculously long (around $10^{65}(M/M_\odot)^3$ yr), so only mini black holes with mountain-type masses of $10^{15}$ g would evaporate in the current age of the universe. There is no evidence for such things existing, although conceivably they might have been formed near the beginning of the universe. This is therefore not likely to be tested in the forseeable future, especially because radiation from other sources (e.g., the accretion of stray molecules) dwarfs Hawking radiation.

**Astrophysical applications of GR and black holes**

A spherical star in general relativity obeys a modified form of the equation of hydrostatic equilibrium, based on the changed gravitational potential and the fact that other forms of energy gravitate. This is called the Oppenheimer-Volkoff or Tolman-Oppenheimer-Volkoff equation, and it is

$$\frac{dP}{dr} = -\frac{G[\rho(r) + P(r)/c^2][M_r + 4\pi r^3 P(r)/c^2]}{r[r - 2GM_r/c^2]} \tag{9}$$

where $\rho(r)$ is now the mass-energy density (not just the rest mass density), but we still have $M_r = \int 4\pi r^2 \rho\, dr$.

Many calculations involving black holes don't go the full GR route of Schwarzschild, let alone including rotation via Kerr. This is because it takes much longer, and often the uncertainties in gas dynamics or MHD overwhelm the inaccuracy in not using real GR. Some calculations just use Newtonian gravity, which is often accurate enough (this is the case for many accretion disk simulations). If more accuracy is desired without grinding your supercomputer to a halt, another approach is to use pseudo-Newtonian potentials that mock up some of the important "pit in the potential" features of GR. One of the best of these is the Paczynski-Wiita potential

$$\Phi = -\frac{GM}{r - 2M} \ . \tag{10}$$

This gets the radius of the ISCO correct, as well as the radius of the marginally bound orbit (where the energy is $e = 1$, so a perturbation outward could allow it to escape to infinity). It also comes close to getting the binding energy right. It has a major advantage over pure Newtonian potentials that it does have a steep drop into the maw of the black hole; therefore, if calculations of behavior near $r = 6M$ or so are important, this is a good compromise.

Finally, let's say a few words about whether black holes actually exist! Their identification poses a difficult problem, because their existence is something of a negative proposition: if nothing else fits, maybe it's a black hole. However, good evidence is to be had from a variety of quarters. Neutron stars can't have $M > 3\,M_\odot$, so if a compact X-ray source in a binary is seen that has a mass greater than this, it's probably a black hole. On a larger scale, dynamical measurements of stellar movements in the cores of several galaxies imply $M > 10^8\,M_\odot$ or so in small region ($\sim$pc). This, too, probably means there are black holes there, although one has to be careful; there was once an announcement of a $10^{11}\,M_\odot$ discovered in this fashion, but it turned out that the core was actually the location of a galactic collision, so the velocities weren't orbital velocities. In any case, there is no virtually no serious opposition to the existence of black holes, although there are still those who have philosophical objections to the ideas of event horizons.