General Relativity

When massive stars die in a supernova, they leave behind a compact remnant: either a neutron star or a black hole. For such objects, Newtonian gravity is no longer accurate enough, and general relativity must be used. Note that black holes, in particular, have had and continue to have a major influence on the universe. It is thought that radiation from the first supermassive black holes may have reionized the universe, accelerating structure formation. In addition, it now appears that light from active galactic nuclei (powered by black holes) may compose 1/3 to 1/2 of all light ever emitted in the universe.

We'll start off with qualitative statements of some of the most important concepts of general relativity, then sketch some of the mathematical formulation.

(1) As in special relativity, space and time are both considered as aspects of spacetime. However, whereas in special relativity spacetime is "flat" (in a sense to be defined later), in general relativity the presence of gravity warps spacetime.

(2) The natural motion of objects is to follow the warps in spacetime. "Matter tells space how to curve and space tells matter how to move." An object that is freely falling (i.e., following spacetime's warps) does not "feel" force, meaning that an accelerometer would measure zero. The path of a freely falling particle is called a geodesic.

(3) The only "force" in this sense that can be exerted by gravity is tidal force. That is, if an object has finite size, different parts of it want to follow different geodesics, and these deviate. Geodesic deviation is the GR equivalent of tidal forces.

(4) Because of this deviation, global spacetime is not flat and there is no coordinate transformation that will make it look flat everywhere.

(5) However, THE most important principle of GR is that in a sufficiently small region of spacetime (small spatial scale, small time interval), the spacetime looks flat. This means that there is a local inertial frame that can be defined in that small patch of spacetime. In that local inertial frame, all the laws of physics are the same as they are in special relativity (electrodynamic, hydrodynamic, strong+weak nuclear, ...)!! This is called the *equivalence principle* (elevator analogy). It means that in practice one of the best ways to do calculations in GR is to do them in the local inertial frame and then use well-defined transformations between the local and the global frame.

(6) All forms of energy gravitate. In the Newtonian limit, rest mass is overwhelmingly the dominant component, but in ultradense matter other forms can be important as well.

(7) A rule of thumb about the importance of GR uses the quantity GM/rc^2, where M is the total mass-energy (possibly including negative contributions from potential energy) and r is some distance. The fractional deviations from Newtonian answers are often of order

GM/rc^2 (not always true, e.g., deflection of light in weak-field limit is twice Newtonian). For the Sun, $GM/c^2 \approx 1.5$ km.

Now let's go into some more detail. Here I'm following Shapiro and Teukolsky, chapter 5, which summarizes some of the astrophysically relevant aspects of general relativity (much of this is taken verbatim from them).

In Newtonian gravity, you can express the field theory as a potential equation:

$$\nabla^2\Phi = 4\pi G\rho_0 \tag{1}$$

where ρ_0 is the rest-mass density. In GR, since all forms of energy gravitate, these must be included as well. In particular, gravitational fields have energy and therefore gravitate. This turns the equation into a nonlinear equation, which can be schematically represented as $F(g) \sim GT$, where g represents the gravitational fields, G is the gravitational constant as before, F is some nonlinear differential operator, and T describes all the nongravitational forms of energy. This equation must of course reduce to the Newtonian form in the weak-field limit. Einstein produced just such an equation, and did so against a geometric backdrop.

This geometry is the geometry of spacetime. Thinking about special relativity, one considers *events*, which are labeled by the time and location of occurrence. Two nearby events in spacetime have, in special relativity, an interval (or pseudo-distance)

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2}$$

as measured by a local inertial observer (i.e., one in a Lorentz frame). There are different choices for the sign of this distance, but that's what I'll use. This interval ds is a Lorentz invariant, meaning that all inertial observers will agree on ds even though they may disagree on dt, dx, dy, or dz. **Ask class:** what is the interval measured if the two events are the locations and times of a given photon? Zero. In fact, $ds^2 = 0$ if and only if the world line of the particle in question is *null*, so that it's a photon or some other massless particle. If we use $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$, we can write this more compactly as

$$ds^2 = \eta_{\alpha\beta}x^\alpha x^\beta \ , \tag{3}$$

where $\eta_{\alpha\beta} = (-1, 1, 1, 1)$ and there is an implicit sum over repeated indices (the Einstein summation convention). The tensor η is the Minkowski tensor, and in general this way of defining distances is a metric approach (no, not cm, kg, etc.!). The Minkowski space is a flat space.

It is of course possible to define other coordinates, such as spherical coordinates, where the diagonal elements are not -1 or 1, and it is also possible to define noninertial coordinates. If the space is Minkowski, however, it is always possible to find a coordinate transformation that brings it back to the Minkowski form (and, therefore, the space is still flat; "If you call a tail a leg, how many legs does a dog have? Four. Calling a tail a leg doesn't mean it is one." A. Lincoln). More generally, in curved spacetime the metric can be expressed as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{4}$$

but *no* change of coordinates can bring it back to the Minkowski form. That is, you can't make a transformation from the coordinate system $\bar{x}^\alpha$ to the coordinate system x^α,

$$g_{\alpha\beta} = \frac{\partial \bar{x}^\lambda}{\partial x^\alpha} \frac{\partial \bar{x}^\sigma}{\partial x^\beta} \bar{g}_{\lambda\sigma} \tag{5}$$

(the general transformation), so that $g_{\alpha\beta} = \eta_{\alpha\beta}$. However, ds^2 is still an invariant. As in SR, the proper time is $d\tau = (-ds^2)^{1/2}/c$.

The principle of equivalence means that we can choose any event in spacetime to be the origin of coordinates, and at that point can diagonalize the metric to the Minkowski form. We can also eliminate the first derivatives of the metric components, so that the interval can be written

$$ds^2 = \left[\eta_{\alpha\beta} + \mathcal{O}(|x|^2)\right] dx^\alpha dx^\beta \ . \tag{6}$$

In this system of coordinates, all laws of physics proceed (within this small region of spacetime!) as if no gravity were present.

The principle of equivalence guides us on how to transform laws of physics in the presence of gravity. The precise transformations are well-defined but more involved than we want at this point. However, an example is the principle of conservation of energy and momentum, which can be written in the form $\nabla_\alpha T^{\alpha\beta} = 0$, where T is the stress-energy tensor (seen before). The partial derivative ∇_α needs to be replaced by something that involves curvature terms, called the "covarient derivative". S+T liken this to the fact that when we use non-Cartesian coordinates, e.g., spherical, the gradient isn't simply $\partial_r A^r + \partial_\theta A^\theta + \partial_\phi A^\phi$. You have to take into account what the coordinates are doing as well.

The equation of motion of a particle on a geodesic, i.e., one not under the influence of nongravitational forces, is (to cut to the chase)

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \ . \tag{7}$$

Here $\dot{x}^\alpha \equiv dx^\alpha/d\lambda$, where λ is *any* parameter along the worldline of the particle (a common choice for a massive particle is τ, the proper time, but this doesn't work for photons). Also, $\Gamma^\alpha_{\beta\gamma} = g^{\alpha\beta}\Gamma_{\alpha\beta\gamma}$ is the Christoffel symbol; this is what is used to construct covariant derivatives. Here $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$ and

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha}) \tag{8}$$

and commas denote partial derivatives. Now, this formidable-looking equation satisfies the equivalence principle, because in a local inertial frames (where, remember, derivatives of metric elements vanish) we have $\Gamma = 0$ (fortunately!), so this equation says there is no acceleration and the particles travel in straight lines.

One can use this formalism to figure out the gravitational redshift in a static gravitational field, i.e., where nothing depends on time. A comparison to the weak-field Newtonian form will then allow an understanding of what some of the spacetime components must be.

Suppose you have light of some frequency. The observed frequency is just the reciprocal of the time between peaks: $\nu_{\rm em} = 1/d\tau_{\rm em}$, where this is measured in the emitter's frame. From the definition of $d\tau$ and ds, this is

$$\nu_{\rm em} = \frac{c}{(-g_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}_{\rm em}} \; . \tag{9}$$

If the emitter is stationary, the space components $dx^i = 0$, and the same is true for the receiver. Since a similar expression holds for the receiver, we have

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{\left[(-g_{00}^{1/2}dx^0\right]_{\rm em}}{\left[(-g_{00}^{1/2}dx^0\right]_{\rm rec}} \; . \tag{10}$$

Here, dx^0 can't vary, because the spacetime is static (nothing depends on the time, which is the coordinate dx^0). Therefore, we have finally

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{(-g_{00})^{1/2}_{\rm em}}{(-g_{00})^{1/2}_{\rm rec}} \; . \tag{11}$$

So what? Well, now we can use this to figure out g_{00} in the weak-field limit by using Newtonian energy conservation. The effective mass of a photon of energy $h\nu$ is $m = h\nu/c^2$. In a Newtonian gravitational field with a potential Φ, the total energy of the photon is $h\nu + m\Phi$. We therefore find, from conservation of energy, that

$$\frac{\nu_{\rm rec}}{\nu_{\rm em}} = \frac{(1+\Phi/c^2)_{\rm em}}{(1+\Phi/c^2)_{\rm rec}} \; . \tag{12}$$

Therefore, $(-g_{00})^{1/2} \approx (1+\Phi/c^2)$.

Well, I'm getting tired of writing c and G all the time! The solution to this adopted by the community is similar to that in QM, where $\hbar = 1$ is a standard substitution. Here we can set $c = G \equiv 1$; when we need units, we can figure out how many factors of G and c go in that way. Incidentally, since gravity is so weak, G is very poorly determined experimentally. One consequence of this is that $GM_\odot$ is known to about eight significant figures, whereas $M_\odot$, in grams, is known to only about four significant figures!