Convection

Chapter 5 in HK.

Focus on aspects of mixing length theory. Present picture first, then general ideas, then more specifics. At the end, students should have an idea of how to calculate with it, but also a solid appreciation of its strengths and weaknesses.

We are now going to tackle the third major type of energy transport, convection. There are aspects of this that differ significantly from our treatment of radiative and conductive energy transport. In those cases it might be difficult in practice to calculate a particular opacity, but in principle it's fairly clear: since microscopic quantum mechanics underlies all of it, one "simply" calculates the equilibrium properties of atoms and electrons for a given set of local conditions.

Convection is different. The general process of convection has some fluid element rising or falling and dissipating its energy as it does so. In its full generality, therefore, convection is a *nonlocal* process and involves very poorly determined details of fluid mechanics. Alas, it is also exceedingly important for most stars, especially the low-mass ones.

As a way around this, "mixing length theory" has been developed. It has the advantage that it mocks up a local treatment of convection, and often doesn't do too badly for real stars. Unfortunately, in stars like the Sun it violates its own assumptions and is laced through with estimates and approximations. Convection is still a major problem in stellar models, and no clear alternative to mixing length theory has arisen yet.

General idea of convection

(draw on board) Suppose you have some element of fluid at a higher temperature than the surroundings. **Ask class:** for pressure balance with its surroundings, what does this mean about the density? Density is less, so buoyancy effects cause it to rise. In mixing length theory the assumption is that the fluid element rises some distance ℓ, then releases its energy (i.e., comes into thermal equilibrium with the new environment). The reverse process can also happen: a cool fluid element, which is denser than average because of pressure balance, sinks some distance ℓ and gains energy from the environment. In both cases, energy is transported outward. Of course, the fluid element can "leak" energy on its way up or down as well.

This is the core of mixing length theory. The amount of energy that it transported depends on ℓ, the buoyant velocity w, the heat content of the star as a function of depth, and how much energy drifts out of the elements as they rise or fall. These quantities can be computed at a single radius, making MLT a "local" theory and therefore useful for computation.

Assumptions of MLT

These assumptions constitute the "Boussinesq" approximation: (keep these on the board).

1. A readily identifiable fluid element has a dimension comparable to ℓ.

2. The mixing length is much shorter than any other scale length associated with the star (e.g., pressure, temperature scale heights). *Violated* for the sun, where the pressure scale height is comparable to what is needed for ℓ!

3. The fluid element is always in pressure equilibrium with its surroundings. This means, for example, that ℓ/v_s is much shorter than ascent or descent times.

4. Acoustical phenomena, shocks, etc., may be ignored.

5. The density and temperature of the fluid element deviates only slightly from the environment.

Together, these assumptions mean that the fluid is almost incompressible and that density and temperature variations are small. Note that assumption 2 is often violated, and that assumption 1 is troublesome by itself: if the fluid element is the same size as ℓ, it is tough to picture the slow drift and leakage of energy!

Another problem is that lab experiments can't simulate the conditions in stars, in the sense that some of the dimensionless numbers that characterize the fluid are a factor of a billion or so different between stars and laboratories. However, note also that this is a common situation in astrophysics. What you have to do in these circumstances is do the best you can, make reasonable assumptions, and solve a simpler problem. It would, of course, make no sense to go wild with this and end up with a finely refined variant of MLT when you know that some of the initial assumptions are suspect!

Simple criterion for convection

Consider an ambient medium of temperature T, pressure P, and density ρ. In it let there be a fluid element of temperature T', pressure P', and density ρ'. Assume that the element has a volume $V \sim \ell^3$, where we're not specifying the exact shape. **Ask class:** what is the buoyancy force on this element? From Archimedes' principle, it is $F_{\rm buoy} = \rho V g - \rho' V g$. Assuming $\rho' < \rho$, the element therefore rises.

We now need to figure out how the temperature of the element changes as it rises, and how fast it rises. Temperature first. Let's define

$$\beta = -\frac{dT}{dz} \qquad (1)$$

as the negative of the temperature gradient ($\beta > 0$ almost always, since z increases towards the surface). We then want to compare this with the way that the temperature of the element changes. This is

$$\frac{dT'}{dz} = T'\frac{d\ln T'}{d\ln P'}\frac{d\ln P'}{dz}\ . \tag{2}$$

Ask class: by assumption, what is the relation between P and P'? Assumed pressure equilibrium, so they're equal.

Assume first that we have no energy exchange as it rises. Then the change in temperature with pressure must describe adiabatic variations. That is, $d\ln T/d\ln P = (d\ln T/d\ln P)_{\rm ad} = \nabla_{\rm ad}$. The last factor, $d\ln P/dz$, is the pressure scale height λ_P. Finally, we simplify by assuming $T' \approx T$ and replacing it with T. This gives the adiabatic change in temperature of the element,

$$\left(\frac{dT'}{dz}\right)_{\rm ad} = -\frac{T}{\lambda_P}\nabla_{\rm ad} = -\beta_{\rm ad} \tag{3}$$

where the "ad" subscript means adiabatic.

Okay, now what happens in different situations? Ask class: suppose that the temperature of the element drops more rapidly than that of the surroundings. Will the element keep rising? No, because it will become denser and sink. Only if the temperature of the element decreases more slowly with increasing z will buoyancy continue to make it rise. Since the temperature gradient is negative, this condition for convective instability (called the Schwarzschild criterion) is

$$\left(\frac{dT'}{dz}\right)_{\rm ad} > \left(\frac{dT}{dz}\right)\ . \tag{4}$$

This can also be written $\beta > \beta_{\rm ad}$, or $\nabla > \nabla_{\rm ad}$, where $\nabla = d\ln T/d\ln P$ and is not the gradient operator. It is also equivalent to the condition that at some point the entropy decreases with increasing radius in the star.

Note that if convection occurs then of course it will have an effect on the temperature structure as well, so this would have to be taken into account.

Accounting for radiative leakage

In the previous treatment, we assumed no heat exchange, but of course in reality there will be some. In a general way, the heat changes by

$$\frac{dQ'}{dt} = -\nabla\cdot\mathbf{F}_{\rm rad}\ , \tag{5}$$

where $F_{\rm rad}$ is the radiant flux out to the ambient medium and Q is the heat content per unit volume. In the Boussinesq approximation we consider density perturbations only when they are coupled to gravity to cause buoyancy. Then the change in the heat content of the parcel is $dQ = \rho c_P dT$ and

$$\left(\frac{\partial T'}{\partial t}\right) = -\frac{1}{\rho c_P}\nabla \cdot \mathbf{F}_{\rm rad} \ . \tag{6}$$

Here c_P is in erg g^{-1} K^{-1}. Using a diffusion approximation for the radiative flux, we have $F_{\rm rad} = -K\nabla T$, and defining the thermal conductivity $\nu_T = K/\rho c_P$ we have

$$\left(\frac{\partial T'}{\partial t}\right) = \nu_T \nabla^2 T \ . \tag{7}$$

This is called Fourier's equation. What we'd really like to do, however, is determine the change in temperature while following the fluid element, so we need a Lagrangian description, and a new gradient operator:

$$\frac{DT'}{Dt} = \left(\frac{\partial T'}{\partial t}\right) + \mathbf{w} \cdot \nabla T' \ , \tag{8}$$

where $\mathbf{w}$ is the velocity. This expression combines two things that affect the temperature of the element: (1) without any motion, heat would diffuse out, and (2) without any loss of heat, the temperature will change because of motion up or down. To squiggle accuracy (and remember, that's all that is justified here because of the monumental assumptions that have gone into the treatment so far), we can replace the Laplacian with $1/\ell^2$ ("cancel the d's in derivatives"), so that $\nabla^2 T = (T - T')/\ell^2$. We also can write $\nabla T' = -\beta_{\rm ad}$. Finally, we can realize that the vertical rate of change of the ambient temperature, DT/Dt, is simply $-\beta w$. Combining these expressions and defining $\Delta T = T' - T$, we find finally

$$\frac{D\Delta T}{Dt} = (\beta - \beta_{\rm ad})w - \frac{\nu_T}{\ell^2}\Delta T \ . \tag{9}$$

Buoyant speed of fluid elements

First crack is to just equate the acceleration with the buoyant acceleration. Then

$$\frac{dw}{dt} = \frac{(\rho - \rho')}{\rho} g \ . \tag{10}$$

But it isn't quite so. **Ask class:** what other force would act on a rising fluid element or bubble? Viscous drag. All else being equal, do we expect a large or a small bubble to rise faster? Large, for the same reason that in an atmosphere big rocks fall faster than small rocks. Bubbles in Baltimore Aquarium, where the largest rises first and fastest. The acceleration goes like the volume, but drag goes like the surface area. The actual drag is like $\nu\nabla^2\mathbf{w}$, and again we can replace the ∇^2 with $1/\ell^2$ to this level of accuracy, finding a refined equation of motion

$$\frac{dw}{dt} = \frac{(\rho - \rho')}{\rho} g - \frac{\nu}{\ell^2} w \ . \tag{11}$$

This additional term decelerates the parcel.

We'd like to relate this to temperatures instead of densities. To do this we make use of the coefficient of thermal expansion at constant pressure

$$-Q = \left(\frac{d\ln\rho}{d\ln T}\right)_P . \tag{12}$$

Using this, we have

$$\frac{(\rho-\rho')}{\rho} = -Q\frac{(T-T')}{T} = \frac{Q}{T}\Delta T . \tag{13}$$

In general Q is a pain in the *** to compute because of its composition dependence, including hairy things like ionization. However, if we make our usual power-law approximations in which $P = P_0\rho^{\chi_\rho}T^{\chi_T}$ then $Q = \chi_T/\chi_\rho$. The equation of motion then finally becomes

$$\frac{dw}{dt} = \frac{Qg}{T}\Delta T - \frac{\nu w}{\ell^2} . \tag{14}$$

Convective Fluxes

As we said before, the convective flux depends on the heat content, the difference in heat between the fluid element and its surroundings, and the speed with which the element rises. Therefore, $F_{\rm conv} = \rho w c_P \Delta T$. For efficient convection (defined in the book, and typical of stars),

$$F_{\rm conv} \approx \frac{\rho c_P \ell^2 T g^{1/2} Q^{1/2}(\nabla - \nabla_{\rm ad})^{3/2}}{\lambda_P^{3/2}} . \tag{15}$$

Since most stellar convection is efficient and adiabatic, one result of this is that the temperature gradient with pressure is very close to the adiabatic value. This flux can also be written using the Mach number of the fluid elements:

$$F_{\rm conv} = \frac{\rho c_P T}{\ell Q g} v_s^3 \left(\frac{w}{v_s}\right)^3 . \tag{16}$$

where v_s is the sound speed and the expression in parentheses is the Mach number. This high dependence is a problem, and if the Mach number approaches unity then yet another assumption of MLT is violated. In practice, people often set 1/2 as the upper limit of the Mach number. Another indication of the down and dirty approach often necessary when dealing with convection. People often parameterize the mixing length ℓ by writing it as $\ell = \alpha\lambda_P$, where α is the mixing length parameter. This parameter is usually of order unity (as for the sun), which means that strictly speaking one of the assumptions (number 2) of the Boussinesq approximation is violated.

Show graph of when outer envelope is and is not convective. Relate to magnetic field strength in stars: high for low-mass stars (flare stars), lower for massive stars. Also, possibly related to observed abundances of some elements such as lithium and some $Z \sim 20 - 30$ elements. As a closing, indicate that there are a number of attempts to do better than

MLT; overshooting, for example. "Correct" treatment would require full fluid (MHD in general) equations, and is very difficult and complicated. When we get to modeling stars, we'll see a bit more how to include convection in reality.