Other radiative opacities and conduction

H^- Opacity

In the atmospheres of cool stars, free-free and bound-free transitions in the negative hydrogen ion, H^-, can be extremely important (it is the most important source of opacity in the solar photosphere, for example). The extra electron is bound, but not by much: only 0.75 eV. **Ask class:** how would we estimate the highest temperature at which this would exist? About when $kT = 0.75$ eV, or around a few thousand degrees. Note also that for this ion to be formed, neutral hydrogen has to capture a free electron, meaning that both have to exist. Therefore, certainly not important when there is no neutral hydrogen, or when the temperature is so low that everything is neutral. Metals, with more loosely bound outer electrons, can contribute to the free electron population and therefore are important for the H^- opacity. At $T < 2500$ K or for low-metallicity stars, the H^- opacity is small. For 3000 K$< T <$6000 K, $10^{-10} < \rho < 10^{-5}$, and reasonable hydrogen and metal abundances, an approximate fitting formula gives

$$\kappa_{H^-} \approx 2.5 \times 10^{-31}(Z/0.02)\rho^{1/2}T^9 \text{ cm}^2\text{ g}^{-1} \ . \tag{1}$$

Tables of this opacity exist, and are to be preferred for serious applications.

When the temperature is below about 3000 K, molecules and dust form and are extremely important for the atmospheric opacities of low-mass stars. **Mention:** water (gaseous, of course!) has been found in sunspots, where the temperature is low(!). Molecular and dust opacities are extremely complicated because of the number of extra degrees of freedom, and are still not especially well understood. For a recent review, see Allard et al. 1997, ARA&A 35, 137: "Model Atmospheres of Very Low Mass Stars and Brown Dwarfs"

Conduction

Here, will follow the discussion in Clayton more than in H+K, because Clayton has a more extensive discussion of conduction.

In the last lecture we discussed energy transport by photons. Now we will talk about a type of energy transport by particles: conduction. In a general way, some of the same principles apply. When there is a temperature gradient, particles from the hotter region can carry energy to the cooler region.

Ask class: when we eventually calculate the conductive opacity, how should it be combined with radiative opacity to form a total opacity? It should be added harmonically, because the lower opacity will determine the transport of energy.

First, then, let's think about which particles will contribute. **Ask class:** in an ideal gas of temperature T, what is the energy per particle? It's just $E = \frac{3}{2}kT$. **Ask class:** Given that, in thermal equilibrium which move faster, electrons or ions? Electrons, because they are lighter and the energy in this situation is $\frac{1}{2}mv^2$. So, the electron velocities are greater than the ion velocities by a factor $(m_{\rm ion}/m_e)^{1/2} \approx 43\, A_{\rm ion}^{1/2}$. **Ask class:** how would this change when the electron gas is degenerate? It would be an even more extreme velocity ratio, because the electrons are forced into higher energy (and hence higher velocity) states. This means that electrons dominate conduction, and that to a reasonable approximation you can think of a fast-moving electron gas and a nearly stationary background of ions.

Now consider what would happen if you have a region of high density and temperature next to a region of low density and temperature (draw). The electrons in the high density/temperature region will move more quickly and with higher electron flux than the ones in the low density/temperature region. Therefore, there is a net flux of electrons. **Ask class:** if this occurred, what would happen to oppose it? Answer is that an electric field would develop due to the charge imbalance, and would quite rapidly prevent any net flux of electrons. Therefore, the flux j_+ of electrons must equal the flux j_- of electrons. **Note:** this is different from radiative case, where there *can* be a net flux of photons. **Ask class:** in this case, then, how can there be a net transfer of energy? Even though particle net flux is zero, energies are not the same, so energy flux is nonzero.

Compare cross sections: consider ionized carbon gas at $T = 10^6$ K and $\rho = 1$ g cm^{-3}. The radiative opacity is dominated by free-free absorption, and the Rosseland mean cross section is $\sigma_{\rm f-f} = 2.3 \times 10^{-21}$ cm^2. Therefore, $1/e$ length for a characteristic photon is 4.5×10^{20} per square centimeter.

These same ions will scatter nondegenerate electrons with the Rutherford differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{mv^2}\right)^2 \frac{1}{(1-\cos\theta)^2} = 5.2\times 10^{-27}\left[\frac{Z}{E({\rm MeV})}\right]^2 \frac{1}{(1-\cos\theta)^2}\ {\rm cm^2 sterad^{-1}}\ . \qquad (2)$$

Diverges! **Ask class:** why does it diverge, and how can this be resolved? Diverges because extreme forward scattering $\cos\theta \longrightarrow 1$ happens for larger and larger distances from the ions. In general, in fact, long-range scattering dominates over short-range scattering for Coulomb. (all logarithms are 10) When the distance of the electron from the ion is 1/2 the distance to the next ion, however, this gets cut off. If this is integrated for $\theta > \pi/2$, we find the $1/e$ number of ions per square centimeter that can backscatter electrons in a single scattering is 3.1×10^{16} ions per square centimeter. That is, the mean free path is 10^{-4} times the mean free path for a photon. For nondegenerate matter in stars, radiation is much more important than conduction.

Note that this form (the prefactor) is also important in Bondi-Hoyle accretion. Note also that the $1/v^4$ in this means that if electrons are accelerated, e.g., by an electric field, they can run away; this can lead to nonthermal populations of electrons. Examples might include the solar corona (where reconnection-type processes produce the electric field) or in accretion disks around magnetized neutron stars).

Qualitative aspects of conduction. **Ask class:** will electron scattering off of electrons be more or less important than electron scattering off of ions? Ions are more important, because they have a larger charge and the cross section goes like Z^2. Now suppose that the electrons are degenerate. **Ask class:** how would you expect the relative opacity to scattering off of ions and electrons to change? Even more extreme, because in degenerate electrons both the initial and final state of the electron has to be unoccupied. **Ask class:** At a temperature T and electron Fermi energy E_F, about what fraction of the electrons can interact? kT/E_F. So, for e-e scattering, need initial and final states for both electrons to be unoccupied. For e-i scattering, need this only for one electron.

Ions can be shielded, reducing cross section. Also, if they are in a perfect lattice, the potential is exactly periodic and the electrons move as if they were free (**Mention:** this is why metals on Earth conduct so well; they are nearly periodic so the electrons move a long way). Therefore, e-i scattering depends on impurities and/or imperfections relative to a perfect lattice. **Reminder:** this is one place where the lack of catalysis to equilibrium nuclear matter is important, because it increases resistivity dramatically! In any case, this is a reason why conductivity is so important in high-density things like neutron stars: in much of the crust, the ions are in a lattice-like structure, so conductivity is huge. **Ask class:** If you had a situation in which the thermal conductivity was large, and you fixed the total energy flux, what would this mean about the temperature gradient compared to when the total conductivity was small? The temperature gradient is smaller when the conductivity is high (equivalently, when the opacity is low). This means that degenerate objects like white dwarfs or neutron stars are essentially isothermal in their interiors.

Now treat conductivity as in Hansen & Kawaler. Can once again use Fick's law of diffusion:

$$F_{\rm cond} = -D_e \frac{dT}{dr} \ . \tag{3}$$

Ask class: why do we write it this way, whereas in radiative diffusion we have instead $d(aT^4)/dr$? Because the energy content of the particles is like kT, whereas the energy density of radiation is like aT^4. It is convenient to recast this in a form that is similar to that for radiation, by defining a "conductive opacity"

$$\kappa_{\rm cond} = \frac{4acT^3}{3D_e\rho} \ . \tag{4}$$

The conductive flux from the diffusion equation is then

$$F_{\rm cond} = -\frac{4ac}{3\kappa_{\rm cond}\rho} T^3 \frac{dT}{dr} \ . \tag{5}$$

Using this, we can see another way how to combine the opacities: the energy fluxes should add, so $F_{\rm tot} = F_{\rm cond} + F_{\rm rad}$, meaning that $1/\kappa_{\rm tot} = 1/\kappa_{\rm rad} + 1/\kappa_{\rm cond}$. Therefore,

$$F_{\rm tot} = -\frac{4ac}{3\kappa_{\rm tot}\rho}T^3\frac{dT}{dr} \ . \tag{6}$$

The diffusion coefficient has the general form

$$D_e \approx c_V v_e \lambda/3 \ , \tag{7}$$

where c_V is the specific heat at constant volume of the electrons (which are degenerate if this is to matter), v_e is the typical velocity of the electrons, and λ is a typical mean free path. Can see why each factor is needed: the higher the specific heat is, the greater the energy difference for a given dT; the higher the velocity is the faster the energy will be transported; and the higher the mean free path is, the greater the temperature gradient that will be sampled.

Now let's go through a rough derivation of the dependences of those three factors for nonrelativistically degenerate electrons. *All of this assumes that the ion positions are uncorrelated with each other*; lattice effects can be substantial in a real star. From chapter 3 in HK, the specific heat for degenerate electrons is proportional to $c_V \propto Tx_f(1+x_f^2)^{1/2}$. **Ask class:** to leading order, what is this expression when the electrons are nonrelativistic? Nonrelativistic means $x_f \ll 1$, so this is something like Tx_f. **Ask class:** how does x_f depend on the number density? Like $n_e^{1/3}$. **Note:** the book talks about ρ/μ_e instead of n_e, but I'm using n_e for simplicity. So, that's the first factor: $c_V \propto Tx_f$.

How about the velocity? **Ask class:** for degenerate electrons, what is the approximate energy of the electrons that can interact? Only those at the top of the Fermi sea, meaning electrons within kT or so of E_F, which for strongly degenerate electrons means $\sim E_F$. That means, for nonrelativistic electrons, that $m_e v_e \approx p_F$. **Ask class:** from the definition of x_f, how does it depend on p_F? It is just proportional, so we find that $v_e \propto p_F \propto x_f \propto n_e^{1/3}$. This is the second factor.

Finally, what about the mean free path λ? **Ask class:** what is the mean free path in terms of the scattering cross section and number density? $\lambda = 1/\sigma_C n_I$, where σ_C is the Coulomb scattering cross section and n_I is the number density of the ions. For the cross section, as a rough approximation the usual estimate is a cross section corresponding to an impact parameter that will "significantly" change the path of the electron, meaning one in which the electrostatic energy is equal to the kinetic energy of the electron. Therefore, the impact parameter s is given by

$$m_e v_e^2 \approx Z_c e^2/s \Rightarrow s \propto 1/v_e^2 \propto n^{-2/3} \ . \tag{8}$$

The cross section itself is $\sigma_C \propto s^2 \propto n^{-4/3}$. Therefore, $\lambda \propto n^{4/3}/n \propto n^{1/3}$, and the diffusion coefficient goes like n. Putting in all the constants and mean molecular weights, a decent approximation to the conductive opacity is

$$\kappa_{\rm cond} \approx 4 \times 10^{-8} \frac{\mu_e^2}{\mu_I} Z_c^2 \left(\frac{T}{\rho}\right)^2 \ {\rm cm^2\,g^{-1}}\ . \tag{9}$$

Just for fun, let's apply the same logic when the degeneracy is relativistic. **Ask class:** how does the specific heat go? Like $Tx_f^2 \propto Tn_e^{2/3}$. **Ask class:** how about the velocity? If it's very relativistic, $v \approx c$. **Ask class:** what about the characteristic impact parameter? Then we want to know when $Z_c e^2/s = E_F \approx p_F c \propto x_f \propto n_e^{1/3}$, so the impact parameter s goes like $n_e^{-1/3}$, the cross section goes like $n_e^{-2/3}$, and the mean free path goes like $1/\sigma n \sim n^{-1/3}$. The diffusion coefficient then goes like $n^{1/3}$ instead of n, and the conductive opacity goes like $\kappa \sim T^2/\rho^{4/3}$. I *think* (but am not absolutely sure) that this is the right expression, but unfortunately it has little use in practice because at densities great enough that degeneracy is relativistic, correlations between ions are usually very significant, causing substantial deviations from this dependence.

Comparison of radiative and conductive opacities

Recalling that to a *very* rough approximation the bound-free opacity is

$$\kappa_{\rm b-f} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} \tag{10}$$

how does this compare to the conductive opacity? Compare typical situations in the center of the Sun and in the center of a white dwarf. Suppose that in the center of the Sun we have $T \approx 10^7$ K and $\rho \approx 100$ g cm^{-3}. Then $\kappa_{\rm b-f} \approx 100$ cm^2 g^{-1}. In the center of a white dwarf you might have $T = 10^6$ K and $\rho = 10^6$ g cm^{-3}. Then $\kappa_{\rm cond} \approx 10^{-7}$ cm^2 g^{-1}, according to this formula (it will probably be even less). The net result is that the conductive opacity in very degenerate material is tiny, so to a decent approximation the whole interior can be considered isothermal.

Tabulated opacities

The simple expressions we've used for the different types of opacities are used to illuminate the physics, and in this respect are quite useful. However, when you get down to real modeling you never use them. Instead, you use the painstakingly compiled opacity tables from places like LANL or LLNL. Much of this relies in turn on atomic transition data, which is itself far from complete; a recent Kurucz paper (astro-ph/0003069) says that half the lines in the solar spectrum are not identified! Still, for most purposes (other than for very cool stars) the tables are adequate.