Radiative opacities

Comments on homework: emphasize again the utility of order-of-magnitude estimates: faster, meaning that it allows you to evaluate more mechanisms with the same amount of work, and the arguments are often more convincing. Have seen many papers in which the underlying argument is actually obscured by too much detail.

Midterm on April 4. Class has decided that it will be closed-book, so the exam will be easier and I will provide you with the necessary constants.

Radiative opacity sources: Ask class: what are the ways in which a photon in a star can interact? Can be done off of free electrons, atoms, molecules, or dust (the last two only for very cool stars). For free electrons, have electron scattering (not an absorption) and free-free (inverse bremsstrahlung, which is an absorption). For atoms, have transitions between atomic levels (bound-bound absorption) and ionizations (bound-free absorption). There are also a variety of other strange processes, such as double Compton scattering, but those can be ignored by comparison.

Scattering from free electrons: Let's start with Thomson scattering. Fairly straightforward, really: for photon energies much less than the mass-energy of an electron, the total cross section is a constant:

$$\sigma_T = 6.65 \times 10^{-25}\,\mathrm{cm}^2\ . \qquad (1)$$

Scattering is not isotropic: the differential cross section for a direction θ away from the initial propagation direction of the photon is $d\sigma/d\Omega \sim (1+\cos^2\theta)$. Thus, photons undergoing Thomson scattering are not exactly doing a random walk, since they have a greater chance of going forward or backward than to the side. Makes virtually no difference to the final result, though. The Thomson cross section is proportional to the mass of the particle to the -2 power, so scattering from ions is almost always negligible.

Pure Thomson scattering is elastic, so photons retain their energy exactly. In reality, there is a little bit of recoil; in Compton scattering, in the frame in which the electron was initially at rest, the photon after scattering has on average changed its wavelength by of order the wavelength of a photon whose energy is the electron rest-mass energy (i.e., the electron Compton wavelength):

$$\lambda_f - \lambda_i = \lambda_c(1-\cos\theta) = \frac{h}{mc}(1-\cos\theta) \qquad (2)$$

where λ_i is the initial, and λ_f the final, wavelength of the photon in the original electron rest frame. In the electron rest frame the photon always loses energy, but in the lab frame the electron, if moving fast enough, can increase the energy of the photon (called inverse Compton scattering). When both the electrons and photons are in isotropic thermal distributions, equilibrium is reached (not surprisingly) when the radiation and electron temperatures are equal.

The recoil effects also modify the scattering cross section somewhat and make it frequency-dependent. For initial photon energy ϵ_i, final photon energy ϵ_f, and a photon direction after scattering that is θ away from the initial direction, the differential cross section in the initial electron rest frame is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - \sin^2\theta\right) , \tag{3}$$

where $r_0 = e^2/m_e c^2$ is the classical radius of the electron. This is the Klein-Nishina formula, and it causes significant deviations only when the photon energy is comparable to the electron rest mass energy. If $x \equiv \hbar\omega/m_e c^2 \ll 1$, then the approximate total cross section is $\sigma \approx \sigma_T(1-2x)$. This is therefore only a big deal above a few hundred million degrees. Scattering is often a minor player for photon energies below $\sim$keV or so, corresponding to $\sim 10^7$ K, because at lower energies absorption processes dominate.

Now, in general Thomson scattering is not the only scattering around. **Ask class:** for example, what makes the sky blue? It's Rayleigh scattering, in which the scattering cross section goes like the frequency to the fourth power. The difference in that case is that the molecules have a resonant frequency that is much greater than the frequency of the light. In general, the cross section goes something like $\omega^4/(\omega^2-\omega_R^2)^2$. Therefore, for things with high resonant frequencies $\omega_R \gg \omega$, the cross section goes like ω^4. For low or zero resonant frequencies, as with free electrons, $\omega_R = 0$ and $\sigma \sim$constant. **Ask class:** can they think of a situation in astronomy where frequency-dependent scattering is important? Dust is a perfect example. Bring up connection to high-z Type Ia supernovae and the requirement for gray dust.

Story of optical depth to San Gabriels!

One last bit of scattering to mention for completeness is cyclotron scattering. This occurs when the electron cyclotron energy $\hbar\omega_c = \hbar eB/m_e c \approx 10(B/10^{12}\,\mathrm{G})$ keV is in the frequency range of interest. The cross section for scattering is $\sim 10^6$ times the Thomson cross section, so in the right circumstances (the atmosphere of a neutron star or of a very magnetic white dwarf) it can have a major effect on the emitted spectrum, but for normal stars the energy is way below the energies of interest and it has a negligible effect.

Bound-bound absorption This is when a bound electron absorbs a photon and consequently makes a transition to a higher bound level. An electron in such a higher level has a finite lifetime to decay Δt. **Ask class:** what does this finite lifetime mean about the energy width Γ of the transition? From the uncertainty principle, $\Gamma = \hbar/\Delta t$. Applying time-dependent perturbation theory (see, e.g., Clayton), the cross section of a transition from initial state s to final state k as a function of energy can be written in the Lorentzian form

$$\sigma(\omega) = \frac{2\pi^2 e^2}{m_e c} f_{\mathrm{ks}} \frac{\Gamma/2\pi\hbar}{(\omega - \omega_{\mathrm{ks}})^2 + (\Gamma/2\hbar)^2} \ . \tag{4}$$

Here f_{ks} is the "oscillator strength". There's a sum rule saying that the sum of the oscillator strengths of all transitions of an electron from a given energy level is 1; caution, though, because emission counts negatively for the sum. Most oscillator strengths are less than or of order 1. Story about my $f = 10^6$! The oscillator strength can be computed from

$$f_{\mathrm{ks}} = \frac{2}{\hbar m_e \omega_{\mathrm{ks}}} \left| \langle k | e^{i(\omega_{\mathrm{ks}}/c)\mathbf{n}\cdot\mathbf{r}} \mathbf{p} \cdot \mathbf{e} | s \rangle \right|^2 \ , \tag{5}$$

where $\mathbf{n}$ is the propagation direction, $\mathbf{p}$ is the momentum, and $\mathbf{e}$ is the polarization. This is nasty, however, so people eagerly take the opportunity to simplify by approximating the exponential as 1, based on the observation that the argument is small, in fact of order $Z/137$ for hydrogenic atoms of nuclear charge Z. This is the dipole approximation. For nuclei, unfortunately, although the size $\mathbf{r}$ is 10^{-4} times the size of an atom, the energies (and hence ω_{ks}) are 10^6 times greater, so you can't do this. However, it's often a good approximation for atoms; nonetheless, keep in mind that for high frequencies or highly ionized atoms it isn't quite as good.

As an example, take the Lyα resonance of hydrogen, where an electron in the 1s state is absorbed into the 2p state. This has an oscillator strength of 0.42 and a peak cross section of 7.2×10^{-11} cm. This is *enormous*, a good 10^{14} times the Thomson cross section. However, it is important over only a very small range in energy. The lifetime is about 10^{-9} s, so the energy width is $\hbar/10^{-9} \approx 10^{-6}$ eV.

Generally, therefore, bound-bound transitions have extremely high cross sections over extremely small regions of the spectrum. Given that the Rosseland mean is harmonically weighted, I would have expected that bound-bound transitions, although producing all of the lines that we see in spectra, would be unimportant in energy transport. People in the 1960s and earlier thought so, too, but it turns out that the total opacity in nonionized portions of the Sun can be increased by a factor of 2 or 3 by bound-bound transitions. This is because with all the transitions available from metals, the wings overlap.

Bound-free absorption

Not sharp like bound-bound, falls off like ν^{-3} above a threshold. Below the threshold there is no absorption, so the total cross section looks like a sawtooth (draw). The cross section for hydrogenic atoms is

$$\sigma_{\mathrm{b-f}} \approx 2.8 \times 10^{29} \frac{Z^4}{n^5 \nu^3} g \ , \tag{6}$$

where $g = g(\nu, n, l, Z)$ is the Gaunt factor and is close to unity, n is the principal quantum number, and Z is the charge of the nucleus. At the hydrogen edge this becomes close to

10^{-17} cm^2, which is much less than the peak of a bound-bound transition but much more than the Thomson cross section. Using the ν^{-3} shape, the two are become competitive for half-ionized hydrogen when the photon energy is about 3 keV.

To get the opacity requires multiplying this by the number of ions in this state per gram. Hansen & Kawaler give the expression

$$\kappa_{\rm b-f} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5}\,{\rm cm}^2\,{\rm g}^{-1}\ , \tag{7}$$

where Z is now the fractional abundance of metals. **Ask class:** what's wrong with this expression, or more correctly, what is the domain of applicability? This expression would predict zero bound-free absorption in pure hydrogen, if used blindly. The real point is that the elements with ionization energies comparable to the temperature will contribute most, and at most temperatures of interest in stars the hydrogen is fully ionized. Therefore, as the frequency of the photons and the temperature go up, one metal/atomic state takes over where another has left off. The net result in the interstellar medium is that at far UV to X-ray energies the opacity is dominated by bound-free absorption, and actually ends up having the approximate form $\kappa \sim \nu^{-8/3}$ because of the relative abundances of elements. This means that at low energies (below around 0.5 keV or so, depending on the column depth) the spectrum appears cut off to us.

This expression for the opacity, with a factor $\rho T^{-3.5}$, is called a Kramers opacity. It also applies to free-free absorption, as we will see, and is a very generally useful form of the opacity for much of the interior of stars.

Free-free absorption

In this process a free electron passes by an ion of charge Z, scatters off of it, and absorbs a photon. This is the inverse process to bremsstrahlung. Conservation of energy means that

$$\frac{p_k^2}{2m_e} = \frac{p_s^2}{2m_e} + \hbar\omega\ . \tag{8}$$

Here, of course, we're assuming nonrelativistic electrons. The presence of an ion is very important. For example, suppose we just want an electron to emit a photon. Then it is impossible to conserve both energy and momentum.

The fundamental reason for bremsstrahlung, or braking radiation, is that as an electron moves past an ion it is accelerated and therefore radiates power:

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}a^2(t)\ , \tag{9}$$

where $a(t)$ is the time-dependent acceleration.

The cross section for free-free absorption around an ion of charge Z is

$$\sigma_{\rm f-f} = 3.7 \times 10^8 \frac{Z^2 n_e g_{\rm f-f}}{T^{1/2}\nu^3}\,{\rm cm}^2\ . \tag{10}$$

In a thermal distribution, we can take the Rosseland mean opacity. For a temperature T, the peak in the blackbody distribution is around $3kT$. **Ask class:** would you expect the Rosseland weighting to give an effective ("weighted average") photon energy greater than or less than this? Greater than, because the Rosseland mean is a *harmonic* mean, implying that lower opacities will get higher weight. Since the cross section drops with increasing photon energy, higher energies will get greater weight and the effective mean photon energy is greater than the energy of the blackbody peak (in fact, about $\langle h\nu\rangle_{\rm eff} \approx 5.8kT$).

A reasonable approximation for the free-free opacity is

$$\kappa_{\rm f-f} \approx 10^{23} \frac{\rho}{\mu_e} \frac{Z_c^2}{\mu_I} T^{-3.5}\,{\rm cm^2\,g^{-1}}\ , \tag{11}$$

where μ_e and μ_I are the mean molecular weights that we've used before, *but* remember that only free electrons contribute!

The Total Radiative Opacity

Ask class: At a given photon frequency ν, should the opacities be added harmonically or linearly? Linearly, and in fact it's a bit like a circuit in series (versus one in parallel, which is like the harmonic mean). The only subtlety is that true absorption processes must be corrected for induced emission:

$$\kappa_{\rm tot} = [\kappa_{\rm b-b}(\nu) + \kappa_{\rm b-f}(\nu) + \kappa_{\rm f-f}(\nu)](1 - e^{-h\nu/kT}) + \kappa_s\ . \tag{12}$$

From this, form the Rosseland mean. **Ask class:** would we expect the Rosseland mean of the sum to be the sum of the Rosseland means? No, in fact this is generally untrue. Only when one component of the opacity dominates (**Ask class** what this means; it would mean that that opacity is *high*) is the Rosseland mean of a component really meaningful.

Show graph of trends. In high-temperature region, electron scattering dominates. As temperature decreases, bound-free and bound-bound (mainly bound-free) become most important at low densities, free-free becomes most important at intermediate densities, and conduction becomes most important at high densities.