Polytropes and opacities

Mean molecular weight: demystify it!

Adiabatic indices: It is often useful, in the spirit of power-law relations between variables, to construct "adiabatic exponents" related to the derivatives of one thermodynamic quantity by another. Let's start with specific heats. In general, a specific heat is

$$c_\alpha = \left(\frac{dQ}{dT}\right)_\alpha \ , \tag{1}$$

where α is fixed. For example, for a monatomic ideal gas, $c_V = \frac{3}{2}k$ per particle and $c_P = \frac{5}{2}k$ per particle. As derived in the book, if we assume a pressure of the form $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$, then the ratio γ of the specific heats is

$$\gamma = \frac{c_P}{c_V} = 1 + \frac{P}{\rho T c_V}\frac{\chi_T^2}{\chi_\rho} \ . \tag{2}$$

This γ need not be the γ we've used in gamma-law equations of state! Confusing, yes, but used enough in the literature that it's worth getting to know the following definitions for different gammas, where the "ad" subscript means that we're taking this along adiabats (slow and reversible, so entropy is constant). Then we have three definitions:

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_{\rm ad} \ . \tag{3}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \left(\frac{\partial \ln P}{\partial \ln T}\right)_{\rm ad} = \frac{1}{\nabla_{\rm ad}} \ . \tag{4}$$

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{\rm ad} \ . \tag{5}$$

Weird definitions, no? They are defined in this way so that for special cases (e.g., ideal gas) they end up being the same number. They aren't completely independent: $(\Gamma_3 - 1)/\Gamma_1 = (\Gamma_2 - 1)/\Gamma_2$. Our gamma-law equation of state turns out to use Γ_3 and, generally, not the others.

For an ideal gas, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3$, and for a pure radiation "gas" they're all equal to $4/3$. If all the gammas are equal to γ and are constant, then the equation of state is very simple along adiabats: $P \propto \rho^\gamma$, $P \propto T^{\gamma/(\gamma-1)}$, and $T \propto \rho^{\gamma-1}$. The form for the pressure as a function of density defines a *polytrope*, which we'll explore more in a minute (not limited to all Γs being equal). Hansen & Kawaler discuss how these adiabatic exponents play out for a few special cases (mixture of ideal gas and radiation; mixture of ideal gas and degenerate gas; inclusion of chemical changes via the Saha equation). Won't discuss in class, but should read through to see what happens.

The point of all these adiabatic exponents is that they play different roles. For example, Γ_3 indicates how the heat content of the gas responds to compression, Γ_1 is involved in the sound speed and dynamics, and Γ_2 is important in determining whether convection occurs.

Polytropes: A very useful simplification along these lines is the use of a *polytrope* for the equation of state. This is simply where the pressure is explicitly a power-law function of density, and is often written as

$$P(r) = K\rho^{1+1/n}(r) \ . \tag{6}$$

Assuming such a form means that, in general, if things like energy transfer or thermal balance were included there would be inconsistencies. The way out in these treatments is to ignore those equations(!) and just use the mass equation and hydrostatic equation. This may seem like a lame solution, but it is part of the general approach to look at simple stuff first, and has proved to be quite useful in many ways. Genuine stellar models for normal stars have been available for quite a while now, but polytropes are still useful for things like evaluating mergers of neutron stars, where the 3-D general relativistic is quite complicated enough without worrying about details of equations of state!

Given this, take the hydrostatic equation

$$\frac{dP}{dr} = -\rho g = -\rho\frac{GM}{r^2} \tag{7}$$

and multiply both sides by r^2/ρ, take the derivative with respect to r, and divide both sides by r^2 to get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \ . \tag{8}$$

This is just Poisson's equation, which we could have derived as well by using $g = d\Phi/dr = GM_r/r^2$ and the hydrostatic equation. Now we put this into dimensionless form; we'd like to be able to get the general structure of a star with a polytropic equation of state without worrying about the actual mass or radius. First, let the density be $\rho(r) = \rho_c\theta^n(r)$, where ρ_c is the central density and $\theta(0) = 1$. Then the pressure is $P(r) = P_c\theta^{1+n}(r)$, where $P_c = K\rho_c^{1+1/n}$. Poisson's equation becomes

$$\frac{(n+1)P_c}{4\pi G\rho_c^2}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \ . \tag{9}$$

Next (and last) we need a dimensionless radius, so we define $r = r_n\xi$, where the scale length for a polytrope of index n is defined as $r_n^2 = (n+1)P_c/4\pi G\rho_c^2$. Finally, Poisson's equation becomes the *Lane-Emden equation*,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \ . \tag{10}$$

Whew! The gravitational energy for a polytrope of index n is

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R} . \tag{11}$$

Analytic solutions exist for $n = 0$ (constant density), $n = 1$, and $n = 5$. Unfortunately, the two polytropic indices we're most interested in are $n = 3/2$, which gives $P \propto \rho^{5/3}$, and $n = 3$, which gives $P \propto \rho^{4/3}$, and neither of them have an analytic solution! So, we'd have to solve the Lane-Emden equation numerically. We'll do that in a few weeks when we talk about putting together more realistic stars.

Energy transfer

Perspective on final goal: As a reminder, we've talked about two aspects of the microphysics, energy generation and the equation of state. The third major bit of microphysics is energy transfer, and once we have that we can in principle construct a model of any star in equilibrium.

Ask class: What are some of the ways in which energy can be transferred? From the particle perspective, can do it via photons, electrons, or neutrinos (**Ask class:** why not ions? Because they're too slow). Neutrinos have such a small cross section that they essentially just act as energy sinks most of the time. **Ask class:** How do we compute the optical depth to neutrinos from the center to the surface of the Sun? Rough estimate is possible by $\tau \approx \bar{n}\sigma_\nu R$, where σ_ν is about 10^{-44} cm^2. At $\bar{n} \approx 10^{24}$ cm^{-3} and $R \sim 10^{11}$ cm, the optical depth is only $\sim 10^{-9}$, so we can just consider any energy in neutrinos as a lost cause for normal stars. In supernovae, the optical depth can be a lot higher; in the region where the supernova shock tends to stall in numerical simulations, the optical depth to neutrinos is about 10^{-2}, but because neutrinos contain $\sim 100\times$ the energy that photons or kinetic energy do, this is a major deal. For us, though, we'll just ignore them.

The next question is **Ask class:** how do we determine, qualitatively, whether photons or electrons are more important for energy transfer? What criterion should we use? Very roughly, the idea is that energy transfer occurs when the particle of interest comes from a high-temperature environment and goes to a low-temperature environment, or the reverse. Thus, there are two things that come in: (1) how great an energy density gradient is typically sampled by the particle, and (2) how rapidly does it carry the energy back and forth? In the interiors of most stars, the cross section for photon scattering is less than for electron scattering, and of course photons move faster than electrons, so photons travel farther, faster, and typically dominate energy transfer there. Caveat about convection. However, Ask class: does photon transfer dominate energy transfer in our normal environment on the Earth? No. A big reason is that for a given temperature T, the energy density in photons in thermal equilibrium goes like T^4, whereas the energy density in ions and electrons goes more like T, so at low temperatures the radiation energy density is small. Therefore, for cool things radiation transport isn't as important as convective or conductive transport of energy.

Radiation in vacuum: For a start, let's think about radiation when there is no matter present. In particular, consider a bundle of rays moving through space. Very generally, Liouville's theorem says that the phase space density, that is, the number per (distance-momentum)^3 (e.g., the distribution function), is conserved. For photons, this means that if we define the "specific intensity" I_ν as ergs per everything:

$$I_\nu\, d\nu = \frac{dE}{dA\, dt\, d\Omega\, d\nu} d\nu \ , \tag{12}$$

then the quantity I_ν/ν^3 is conserved in free space. The source of the possible frequency change could be anything: cosmological expansion, gravitational redshift, Doppler shifts, or whatever. The integral of the specific intensity over frequency, $I = \int I_\nu\, d\nu$, is proportional to ν^4.

This is an *extremely* powerful way to figure out what is happening to light as it goes every which way. The specific intensity is all you need to figure out lots of important things, like the flux or the surface brightness, and in apparently complicated situations you just follow how the frequency behaves. Many people don't use this, and their derivations are often overly complicated and subject to error as a result. A current example relates to gamma-ray bursts. The most popular model for the "afterglow" in X-rays, optical, IR, and radio is that there is a blast wave produced by a central explosion, and what we are seeing is radiation from the surface of this highly relativistic blast wave, which in addition could be a jet instead of being spherically symmetric. The quantities of interest (e.g., the light curve of the burst, the flux, etc.) are all derivable from the specific intensity. Given that there is a cosmological redshift ($z \sim 1$ in many cases) and a strong Doppler shift (Lorentz factors $\gamma \sim 300$ are inferred), this can be a very tough road to hoe otherwise. I've also used this extensively in computations of ray tracing around rotating neutron stars, where in general the spacetime is quite complicated.

Note that one consequence of this is that if the frequency doesn't change, the surface brightness is conserved; can also get this from geometric arguments (flux down by r^{-2}, but $d\Omega$ also down by the same factor). Note, however, that surface brightness integrated over frequency would drop like $(1+z)^{-4}$ for cosmological sources.

Effects of having matter around: Ask class: what are the effects of matter? Matter can emit and can absorb, so the specific intensity in any given direction can be altered by its presence. This means that if we define the frequency-integrated specific intensity at a colatitude θ (we assume azimuthal symmetry for simplicity and to correspond to Hansen & Kawaler) as $I(\theta)$, then if the matter in some ds distance in direction θ emits energy at a rate $j(\theta)$ erg s^{-1} g^{-1} then the change in the specific intensity at that point is

$dI(\theta) = j(\theta)\rho\, ds$. Matter can also absorb, in which case if the opacity is κ then the amount taken out of the beam is $dI(\theta) = -\kappa\rho I(\theta) ds$. Combining these gets the equation of transfer,

$$\frac{1}{\rho}\frac{dI(\theta)}{ds} = j - \kappa I(\theta)\ . \tag{13}$$

If we define the source function $S_\nu(\theta) = j_\nu/\kappa_\nu$, realize that the optical depth is $d\tau_\nu = -\kappa_\nu\rho\, dr$ (where dr is in the radial direction, which is at an angle θ to ds), then a little algebra gives

$$\mu\frac{dI_\nu(\tau,\mu)}{d\tau_\nu} = I_\nu - S_\nu \tag{14}$$

as the equation of transfer. Here $\mu = \cos\theta$ and is not related to the other μs we've had around (chemical potential, mean molecular weight... you'd think with two full alphabets we could keep things unique, but no).

Diffusion and random walks: It would be wonderful to be able to treat the motion of photons as if it were a random walk, independent of frequency. Of course, a problem with this is that photons of different energies have different opacities, so it isn't completely straightforward. For the moment, however, let's ignore this and assume that the opacity is independent of the frequency ("gray atmosphere" approximation).

If the photons were to simply scatter, then their progress would be a random walk. Suppose that in step 1 the photon travels a distance and direction $\mathbf{r}_1$, in step 2 $\mathbf{r}_2$, and so on, in random directions but with the same distribution of distances. The mean location after N scatterings is zero (it has to be, due to the isotropy), but the mean square distance is

$$\langle \mathbf{R}^2\rangle = \langle \mathbf{r}_1^2\rangle + \langle \mathbf{r}_2^2\rangle + \langle \mathbf{r}_3^2\rangle + \ldots + 2\langle \mathbf{r_1}\cdot\mathbf{r_2}\rangle + 2\langle \mathbf{r_1}\cdot\mathbf{r_3}\rangle + \ldots \quad . \tag{15}$$

The cross terms all average to zero, but the squared terms are all the same and are the mean square distance traveled in a single scatter. The net result is that the mean square distance after N steps is Nl^2, where l^2 is the mean square distance for a single scattering. Therefore, we get the result that the average distance from the origin is $\sqrt{N}l$. Similarly, if the optical depth to escape is $\tau \gg 1$, the photon typically undergoes $\sim \tau^2$ scatterings before it escapes. If the medium is optically thin, $\tau \ll 1$, of order τ scatterings on average are required. For most rough estimates you can use just $\max(\tau, \tau^2)$ for the number of scatterings. In stellar interiors, $\tau \gg 1$ so we're in the random walk limit and can treat the process of radiation transfer as diffusion.

Now, for pure elastic scattering (Thomson scattering), the photons don't change energy. However, for scattering with recoil (Compton scattering) the photons do change energy, and for absorption and emission the photon loses its identity, so we'd like a somewhat more general way to treat things. We also would like to incorporate the fact that the opacity will in general depend on frequency. The most general way to do this would be to calculate the emission and opacity (or their ratio, the source function) at each point in the star as a function of frequency and solve the transfer equation that way. However, there's a simplification that saves a lot of work and comes up with remarkably good answers.

Rosseland mean opacities: The idea is that LTE is such a good approximation that we can pretty much assume Planck functions and the like everywhere in the star, where over large distances the temperature will change. We then want to know an "average" opacity such that if we assume that the opacity at all frequencies is this average, we'll get the radiative transfer about right. The way to do this is to weight the average harmonically by the Planck function:

$$\frac{1}{\kappa} = \left[\int_0^\infty \frac{1}{\kappa_\nu}\frac{\partial B_\nu}{\partial T}\,d\nu\right]\left[\int_0^\infty \frac{\partial B_\nu}{\partial T}\,d\nu\right]^{-1} \tag{16}$$

where $\frac{\partial B_\nu}{\partial T} = (2h\nu^3/c^2)/(e^{h\nu/kT}-1)$ is the Planck function. This is the Rossland mean opacity. **Ask class:** why is this harmonically weighted? Because the *low* opacities will dominate the transfer, since photons at those energies will go farther and sample larger gradients in temperature. With this definition, the flux is

$$F(r) = -\frac{c}{3\kappa\rho}\frac{d(aT^4)}{dr}\ . \tag{17}$$

Gray atmosphere: Won't go into stellar atmospheres much in this course, but it is useful to know that for many purposes the gray atmosphere solution (using the Rosseland mean opacity) is remarkably good. In this solution, if τ_R represents the Rosseland optical depth from the surface, the temperature as a function of depth is given by

$$T^4 \approx \frac{3}{4}T_{\rm eff}^4(\tau_R + 2/3)\ , \tag{18}$$

where $T_{\rm eff}$ is the effective temperature, such that the radiated flux from the surface is $\sigma T_{\rm eff}^4$. This isn't perfect, of course, but it's surprisingly accurate even when there are lines and edges and stuff. The more important difference from gray atmospheres that exists in real atmospheres is that the emergent spectrum is usually much different (gray dust and Type Ia supernovae).