Thermonuclear fusion, continued

With this lecture, extremely important to have 5 minutes at the end to recap the main points!

Nomenclature: A reaction of the form

$$a + X \to Y + b \tag{1}$$

is often represented $X(a,b)Y$. An intermediate state of an excited nucleus is represented with an asterisk: Z^*, or a second type of excited nucleus could be Z^{**}. There might be such an intermediate state, as in

$$p +^{11} B \to^{12} C^* \,. \tag{2}$$

Here the number ^{11} in front of B means the atomic weight. The excited state, or compound state, often has many ways in which it can break up. For example, $^{12}C^*$ can break up into $^{12}C^{**} + \gamma$ (a lower excited state plus a photon), $^{11}B + p$ (the original combination of boron and a proton), $^{11}C + n$, and so on. In all cases, when considered in isolation the reactions must conserve **Ask class:** (1) energy, (2) linear momentum, (3) charge, (4) baryon number, and (5) lepton number (separately for electron, muon, tau). For these last two, antiparticles count as -1. Angular momentum must also be conserved, but that's more detailed. For example, for neutron decay in free space, $n \to p + e^- + \nu$, the neutrino must be an electron antineutrino to conserve lepton number. The reaction $\gamma \to \gamma + \gamma$ in free space satisfies these conservation laws, but is kinematically forbidden because the two photons must be precisely parallel. Remember that Fermi energy must be included in these equations! **Mention: conservation of energy related to invariance with time; conservation of linear momentum related to homogeneity; conservation of angular momentum related to isotropy. GR has no global energy conservation; time now not same as near big bang!**

For each mode i, can define a mean lifetime for decay τ_i. Therefore, through uncertainty principle, have a width $\Gamma_i = \hbar/\tau_i$ in energy. Thus, more long-lived the process is, the less likely it is. The probability of decay through a particular channel i is then $(1/\tau_i)/\sum_j(1/\tau_j) = \tau/\tau_i$, where $\tau = 1/\sum_j(1/\tau_j)$ is the total mean lifetime. This can therefore also be written $P_i = \Gamma_i/\Gamma$.

The Γ_i can be broken into two factors: a probability that the state is composed of the desired constituents, which is represented by the "reduced width" γ_i^2, and a Coulomb penetration factor, which allows those constituents to get to infinity. Usually written $\Gamma_i = 2P_\ell\gamma_i^2$.

Cross sections.—For some reaction $X + \alpha \to Y + \beta$, we can define a cross section $\sigma_{\alpha\beta}(v)$ as the number of such reactions per second per target divided by the incident flux of the projectiles. Then the reaction rate per volume is $\sigma_{\alpha\beta}(v)vn_\alpha n_X$ cm^{-3} s^{-1}. If the target and projectile are the same, have to divide by two to avoid double-counting. To get the average rate, have to integrate $\sigma_{\alpha\beta}v$ over the particles, which are distributed in a Maxwell-Boltzmann distribution. In center of mass frame, the relative velocities are also distributed that way, so can get the average as in eqn. (6.26) of HK:

$$\langle\sigma v\rangle_{\alpha\beta} = \left(\frac{8}{\pi m}\right)^{1/2}(kT)^{-3/2}\int_0^\infty \sigma_{\alpha\beta}(E)e^{-E/kT}E\,dE\ . \tag{3}$$

The cross section can be written in the form (6.28):

$$\sigma_{\alpha\beta}(E) = \pi\bar{\lambda}^2 g\frac{\Gamma_\alpha\Gamma_\beta}{\Gamma^2}f(E)\ . \tag{4}$$

Factors break down as follows: $\bar{\lambda}$ (which should really have the bar across the lambda, not on top of it) is the reduced DeBroglie wavelength, so that $\pi\bar{\lambda}^2 = \pi\hbar^2/2Em$. This, roughly, is what the cross section would be for a perfect absorber. g is a statistical factor. $\Gamma_\alpha\Gamma_\beta/\Gamma^2$ is the probability that the initial particles will get together multiplied by the probability that the end products will be as specified. The final factor, $f(E)$, is the "shape factor" and depends on the process: can be "resonant" or "non-resonant". Resonant is peaked at some energy (see 6.30 in HK), non-resonant either is intrinsically non-resonant or is well below the peak.

Non-resonant: For this, the barrier penetration factor is $P_l(E) \propto e^{-2\pi\eta}$, where $\eta = Z_\alpha Z_X e^2/\hbar v$, or $0.1574 Z_\alpha Z_X(\mu/E)^{1/2}$ (μ is the reduced mass, $\mu = m_\alpha m_X/(m_\alpha + m_X)$). Putting this together, the cross section for non-resonant interactions is like

$$\sigma_{\alpha\beta} = \frac{S(E)}{E}e^{-2\pi\eta}\ . \tag{5}$$

Net result is that $\langle\sigma v\rangle$ is proportional to $\int_0^\infty S(E)\exp\left[-\left(\frac{E}{kT}+\frac{b}{E^{1/2}}\right)\right]dE$, where $b \equiv 0.99 Z_\alpha Z_X \mu^{1/2}$ (MeV)$^{1/2}$. Clearly, there is a tradeoff between barrier penetration (increasing with increasing energy) and number of available particles (decreasing with increasing energy). The maximum of the integrand is called the "Gamow peak": $E_0 = 1.22(Z_\alpha^2 Z_X^2 \mu T_6^2)^{1/3}$ keV. Full width at $1/e$ max is $\Delta \approx 2.3(E_0 kT)^{1/2}$. For example, at 2.2×10^7 K and using the $p+{}^{12}C$ reaction, $E_0 = 30.8$ keV and $\Delta = 17.6$ keV. This reaction is extremely sensitive to temperature: the peak at 2.2×10^7 K is 5x the peak at 2.0×10^7 K.

The shape factor $S(E)$ has to be determined experimentally. Over the range of interest it is close to constant. Integration of this expression isn't possible in closed form. One trick is to integrate a Gaussian that is of the same height and curvature at maximum, and this works reasonably well because the integrand is strongly peaked.

Point out that the reactions in the Sun are extremely rare at the given temperature. How do we know? Figure out amount of hydrogen that is being converted to helium per time, relate this to the number of possible interactions. Not much at all. Extrapolation to much lower energies, but we still have some confidence because the theoretical curve fits well. Still want to be alert, though.

Bring them back to the point that these reactions are inherently exponential in the temperature, but that it is analytically convenient to express things in power law form. Therefore, in the form $\epsilon = \epsilon_0 \rho^\lambda T^\nu$ that we used before, the logarithmic derivatives are $\lambda = 1$ and $\nu \approx 14(Z_\alpha Z_X \mu)^{1/3}/T_6^{1/3} - 2/3$. For example, $^{12}C(p,\gamma)^{13}N$ at $T = 2 \times 10^6$ K has $\nu = 16$.

Resonant reactions: Near the peak energy of a resonant reaction, the cross section is dominated by the $[(E - E_r)^2 + (\Gamma/2)^2]^{-1}$ Lorentzian form. Integration of this is relatively straightforward because of the peaked nature of the cross section. The net temperature exponent is $\nu = 11.61 E_r/T_9 - 3/2$.

Neutron capture cross sections: **Ask class what the major difference is (no Coulomb barrier!)**. Not important for energy production, but it is important for figuring out how heavy elements got to be the abundances they have. s process: capture is slow compared to beta decay. r process: capture is fast, so more neutron-rich elements can be produced (only in supernovae).

Electron screening: a cloud of electrons will gather around the positively charged nuclei, reducing the Coulomb barrier and increasing the reaction rate. For small corrections of this type, this ends up increasing the rate by a factor $\exp(U_0/kT)$, where U_0 is a screening energy. For protons onto ^{12}C at 2×10^7 K, the factor is $\exp(U_0/kT) \approx 1.25$.

Pycnonuclear reactions: at high densities, nuclear wavefunctions overlap enough that even at zero temperature fusion can occur. This requires densities only found in a white dwarf or neutron star, of order 10^{6-9} g cm^{-3}. Very temperature insensitive. Can also think about this as Fermi energy substituting for temperature.

Stability versus instability?