Dimensional Analysis and Homology Relations

Introduce energy balance (eqn. 3) and energy transfer (4) equations first. Comment on composition etc. (see below). Then introduce idea of homology. Idea: simple spherical stellar models in complete equilibrium, with same uniform composition and microphysics. Then, assume that r and M_r in a star of radius R and mass M are related to each other, with respect to a reference star of radius R_0 and mass M_0: if

$$r = \frac{R}{R_0} r_0 \tag{1}$$

then

$$M_r = \frac{M}{M_0} M_{r0,0} \ . \tag{2}$$

That is, the mass interior to r in the general star is equal to M/M_0 times the mass interior to $r0$ in the reference star.

Mention: problems I had with this kind of argument, which seems a priori ridiculous. Ends justify the means. Power laws are your friends, so do this if possible!

Now write down the four fundamental equations of stellar structure in Lagrangian form.

1. Mass equation, for homologous stars:

$$\rho = \rho_0 \frac{dM_r}{dM_{r,0}} \frac{1}{dr/dr_0} \left(\frac{r}{r_0}\right)^{-2} = \rho_0 \left(\frac{M}{M_0}\right) \left(\frac{R}{R_0}\right)^{-3} \ . \tag{3}$$

2. Hydrostatic equilibrium

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \ . \tag{4}$$

This codifies dynamic equilibrium.

3. Energy balance, or energy equation

$$\frac{dL_r}{dM_r} = \epsilon \ . \tag{5}$$

Here ϵ is the energy generation rate per mass, and this indicates that there is no pile-up in energy (as must be true in steady-state equilibrium; difference between "static" and "steady"). Often parameterized as $\epsilon = \epsilon_0 \rho^\lambda T^\nu$. Note that this neglects composition, and that T dependence is really much closer to exponential because that's the way fusion works. Ask class to think of things (like composition) that are not explicitly included, and for them to think about the T dependence.

4. Transfer equation. Complicated in general, but can simplify because of short mean free path (leads to diffusion). General idea of diffusion is that flux is proportional to the gradient of some quantity, and in our case we have (in Lagrangian form)

$$L_r = -\frac{(4\pi r^2)^2 c}{3\kappa} \frac{daT^4}{dM_r} \tag{6}$$

where the opacity κ is often written as $\kappa = \kappa_0 \rho^n T^{-s}$. Define this as cm^2 g^{-1}. Should know that for electron scattering $n = s = 0$ (ask). Kramers' law $n = 1$, $s = 3.5$.

Now derive the equations relating the derivatives of log quantities to each other, using this Lagrangian formalism. We'll divide out the common factor $d\ln M$. Start with mass equation. Ask class: do we need to retain constant factors like G, c, R_0, and M_0, given that we are going to take the log and then the derivative? No, because these become zero.

1. Mass equation. $\rho R^3 \propto M$. Take log, then derivative, and get $d\ln\rho + 3d\ln R = d\ln M$. Dividing out, we get $d\ln\rho/d\ln M + 3d\ln R/d\ln M = 1$. That means that if we assume power-law relations in the form $R \propto M^{\alpha_R}$ and $\rho \propto M^{\alpha_\rho}$, we get $\alpha_\rho + 3\alpha_R = 1$.

2. Equation of hydrostatic equilibrium. Integrate, drop constant factors, and find $P \propto M^2/R^4$, where P is the central pressure. Assume the pressure can be written in the form $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$ (this is the equation of state) and that χ_ρ and χ_T are the same for all the stars in the set. Taking the

logarithmic derivative, we get

$$4d\ln R + \chi_\rho d\ln\rho + \chi_T d\ln T = 2d\ln M\ . \tag{7}$$

Having done this it should not be necessary to do the rest. It should be possible to simply say we're also taking $T \propto M^{\alpha_T}$ and $L \propto M^{\alpha_L}$. Just in case, however:

3. Energy equation. Integrate, get $L \propto \epsilon M$, so that $d\ln L - \lambda d\ln\rho - \nu d\ln T = d\ln M$.

4. Transfer equation. Integrate, drop constants, get $LM \propto R^4T^4\kappa^{-1}$, so that the logarithmic derivative equation is $d\ln M = -d\ln L + 4d\ln R + (4+s)d\ln T - nd\ln\rho$.

Write up final matrix as in H+K. Don't solve or put in the solution, have students look it up. Mention (but do not do) convection. Write down R and L for CNO burning MS stars, do not derive (again, suggest book).

Observables of Stars, Conventions

What can be directly observed for a distant star? Total brightness, spectrum. For close stars, distance can be measured with parallax or more indirect methods. Introduce idea of spectral bands: B, V, R, I, and so on. Complications of different filters! (but don't confuse them too much with this; bring along Harwit). Apparent magnitude: bands relative to Vega, which is defined as magnitude 0 in all bands. Logarithmic system: 5 magnitudes equals a factor of 100 in flux in a given band. If a star is *dimmer*, its magnitude is *greater*; therefore, something with magnitude +5 in the V band is 100 times dimmer than Vega. Historical antecedant is probably that people would talk about "a star of the first magnitude" being brighter than one of the second magnitude, and probably the proportions between magnitudes are about the level distinguishable by eye. Dark site, good vision, magnitude 6 is the typical naked eye limit. Brightest star in night sky is Sirius, at magnitude -1.5.

Absolute magnitude: magnitude that star would appear to have if it were 10 pc from us (make sure they know what a parsec is!). The Sun has an absolute V magnitude of 4.8. From this, ask the class to calculate its apparent V magnitude (obviously they need the Sun's distance from us and the inverse square law). Also: given that Sirius has an apparent magnitude of -1.5 and an absolute magnitude of 1.4, how far away is it? (answer: 8.6 lyr, or 2.6 pc). Note that this is one way of determining the distance of an object: knowing its true and apparent brightnesses. Danger, though, of complications. See if the class can get idea of reddening, dust, or other stuff.

Bolometric luminosity versus luminosity in different bands.

Spectral types: OBAFGKMRNS. Don't worry so much about details, but general idea is that the "earlier" (towards O) the type is the hotter it is, and the "later" (towards M) the cooler it is. This earlier/later nomenclature is a relic of the idea that this actually represented an evolutionary sequence. Subdivisions within this: 0 through 9 (e.g., G2, the Sun). May also want to introduce luminosity classes: I is most luminous, VI is least luminous ("dwarfs" like the Sun are luminosity class V).

H-R diagram was a consequence of plotting observables against each other. This is often a good idea in general: if the result is a scatter plot, then no correlation, but if the result is a tight line in the plane, you may have something. This was the case in the H-R diagram.

For H-R diagram, take a good amount of time to discuss possible observational problems (for example, dust can change color) and selection effects, as well as assumptions necessary in converting from the observers' diagram (color-magnitude) to the theorists' diagram (temperature-luminosity). Bring up brightness temperature issues in, for example, pulsars.

At end, come back to our theoretically derived relations of radius and luminosity to mass. Show that they work reasonably well, so we have confidence in them.

Time on the main sequence: about $10^{10}(M/M_\odot)^{-2.5}$ yr. This is a lot less than if you converted all the H in the Sun into He, but that's a topic probably best left to later.

If time: Distance determinations to globular clusters using the H-R diagram.