The Virial Theorem

This is a theorem relating the kinetic energy (or temperature) of particles to their potential energy.

Suppose we consider particles that are noninteracting and moving in circular orbits. What relation is there between kinetic and potential energy? Answer: for a particle of mass m in a circular orbit of radius r around a mass M, the potential energy is $-GMm/r$ and the velocity is $\sqrt{GM/r}$ so that the kinetic energy is $\frac{1}{2}GMm/r$. Therefore, $2U = -\Omega$, where U is kinetic energy and Ω is potential energy. This means that the total energy is $U + \Omega = \Omega/2 < 0$.

This already tells us something highly important about gravitation: when things lose energy, they speed up and increase in temperature. That means that gravitationally bound systems have, in this sense, negative specific heat! This is a fact of great importance to understanding such systems. This form of the theorem can be shown to apply as a time average for any system with bounded motion (Landau and Lifshitz).

Mention: globular clusters, gravithermal catastrophe.

Use this to estimate the average temperature in the Sun, assuming that the term U is related to the temperature. The velocity is $\sqrt{GM/r}$, or $v = 4.3 \times 10^7$ cm s^{-1}, so the temperature should be about $kT = \frac{1}{2}m_p v^2$, or $kT \approx 10^{-6} m_p c^2 = 10^3$ eV $= 10^7$ K! Let's do a calculation for the Earth. In that case, $v = \sqrt{GM/r} = 8 \times 10^5$ cm s^{-1}, so $kT = 4000$ K. Well, in both cases this is close to the *average* temperature.

Mention: 10^7 K for Sun is an important result; we'll find later that something like this ($> 10^6$ K, say) is necessary for fusion.

But wait: the Sun certainly isn't a bunch of noninteracting particles going in circles! If so, the *outer* portion of the Sun would be at 10^7 K, which it certainly isn't. Need a more general derivation.

Go through version of Hansen+Kawaler derivation (*do* show explicitly that $\sum F_i \cdot r_i$=U). Ask (before showing) the class to identify the moment of inertia term. After going through this derivation, in which only gravitational forces are considered, act worried that I've been sloppy by not including other forces. Ask the class about what other contributions might come in. Then, lead them to realize that everything else is short-range (strong and weak are inherently short-range; EM isn't, but bulk matter is neutral for that reason).

Mention: remove all charges from one hair, place on floor at feet, force would squash you flat, 1000x more than a NS!

Mention: this is why gravity dominates at large scales despite being by far the weakest at small scales.

Now, ask them under what conditions will the moment of inertia term play a significant role: only when the bulk velocity is a significant fraction of free-fall (if the velocity is ϵ times free-fall, the correction is of order ϵ). Do we have to worry about this? Ask for Sun, for Cepheids. Point out that again, it depends on the accuracy you want. $I = mr^2$ for single particle, $\ddot{I} = 2m\dot{r}^2 + 2mr\ddot{r}$. For radial free-fall, $\dot{r} = -\sqrt{2GM/r}$, so $\frac{1}{2}\ddot{I} = 2GM^2/R - GM^2/R = GM^2/R$, which is the magnitude of the sum of the terms on the right. Therefore, only comparable if $\dot{r}$ is comparable to free-fall.

State $\int_{\mathcal{M}} \frac{3P}{\rho} d\mathcal{M}_r + \Omega$ form, refer to H+K for derivation. However, qualitatively justify the relation with the pressure as in H+K.

Relativistic corrections? How about for the outer portion of the Sun? If there are any, they would be of order β^2. One way that we can do this is to use the first example above to indicate that the thermal velocity will be about equal to the Keplerian orbital velocity. That would be $\sqrt{GM/r}$, or 4.3×10^7 cm s^{-1}. This gives a correction of about 10^{-6}. Indicate importance of $T = 4 \times 10^9$ K for electrons, $T = 8 \times 10^{12}$ K for nucleons (have class derive both). Note that in the center of the Sun, sharing of energy between electrons and protons means $T = 1.5 \times 10^7$ K for both, so 0.3% corrections for electrons; can be substantial for more massive stars late in evolution.

Kelvin-Helmholtz time scale. H+K give the details for a γ-law EOS. Refer students to the book for such details, but use instead the simple estimate of $t_{\rm KH} = GM^2/(LR)$, or about 3×10^7 yr for the Sun. Discuss history of this, how Kelvin used this as an argument against evolution (K-H time for the Earth is similarly small).

Now follow H+K 1.3.1: introduce γ-law EOS and derive total energy. Emphasize important properties of $\gamma = 4/3$. Ask class for what would happen if $W > 0$ (star could unbind itself). Foreshadow $\gamma = 4/3$ relation to relativistic EOS (e.g., radiation).

Mention: here, talk about Zweibel problem, say on the side that the $M > M_J$, small $P_{\rm orb}$ results are selection effects, ask them to think of such effects.

Dynamical time scale. How long would it take for the Sun to react to a sudden and dynamic change (e.g., removing pressure support?). We will have calculated this earlier: it's proportional to $1/\sqrt{G\bar{\rho}}$, or about an hour for the Sun.

Second dynamical time scale. How long will it take for a sound wave to travel across the Sun? Follow H+K derivation, introducing $\Gamma_1 = \left(\frac{d\ln P}{d\ln\rho}\right)_{\rm ad}$. Be sure to point out the role of $\Gamma_1 = 4/3$; mention result quoted in H+K, then ask class what it means. (It means that a star with such an equation of state will not react to a sound wave in the same way; for example, if you squeeze such a star then it stays squeezed).