ASTR 688M
GR practice problems, with solutions

1. A particle is in a circular geodesic at radius r around a star of mass M. Assuming the Schwarzschild spacetime, what is the linear azimuthal velocity of the particle as measured by a local static observer? Recall that the angular velocity as seen at infinity is $d\phi/dt = (M/r^3)^{1/2}$.

Answer:

The local linear azimuthal velocity is $d\hat{\phi}/d\hat{t}$, or $u^{\hat{\phi}}/u^{\hat{t}}$. Using the transformation matrices,

$$v^{\hat{\phi}} = \frac{u^{\hat{\phi}}}{u^{\hat{t}}} = \frac{e^{\hat{\phi}}{}_{\phi}u^{\phi}}{e^{\hat{t}}{}_{t}u^{t}} = \frac{ru^{\phi}}{(1-2M/r)^{1/2}u^t} = \frac{r}{(1-2M/r)^{1/2}}\left(\frac{M}{r^3}\right)^{1/2} = \left(\frac{M}{r-2M}\right)^{1/2} . \quad (1)$$

For example, at the innermost stable circular orbit ($r = 6M$), $v^{\hat{\phi}} = 1/2$.

2. If a star is rotating slowly, its external spacetime in the rotational equator ($\theta = \pi/2$) can be approximated to first order in the rotation parameter $j = a/M$ as the line element

$$ds^2 = -d\tau^2 = -(1-2M/r)dt^2 + \frac{1}{1-2M/r}dr^2 + r^2(d\theta^2 + d\phi^2) - \frac{2jM^2}{r}(dt\, d\phi + d\phi\, dt) . \quad (2)$$

The last term is written explicitly to show that it is symmetric. This means that the diagonal covariant components (g_{tt}, g_{rr}, etc.) are unchanged, but that $g_{t\phi} = g_{\phi t} = -2jM^2/r$. The diagonal contravariant components g^{tt}, g^{rr}, etc. are also unchanged from their Schwarzschild expressions. The off-diagonal contravariant components are $g^{t\phi} = g^{\phi t} = -2jM^2/[r^3(1-2M/r)]$. When converting to a local orthonormal frame (in which the metric is $\eta_{\alpha\beta} = (-1, 1, 1, 1)$), the transformation matrices are the same as before, with the addition of $e^{\hat{\phi}}{}_{t} = -2jM^2/r^2$ and $e^{\phi}{}_{\hat{t}} = 2jM^2/[r^3(1-2M/r)]$.

(a) A particle moves in an equatorial circular orbit with zero angular momentum. What is its specific energy as measured at infinity?

(b) What is its specific energy as measured in its rest frame?

(c) What is its angular velocity as seen at infinity?

Answer:

(a) As always, $u^2 = -1$. Since this is an equatorial orbit, $d\theta = 0$. Since this is a circular orbit, $dr = 0$. Therefore, $u^2 = -1$ reduces to

$$u^t u_t + u^{\phi}u_{\phi} = -1 . \quad (3)$$

But we are given that $u_\phi = 0$, so

$$\begin{aligned} u^t u_t &= -1 \\ (g^{t\alpha} u_\alpha) u_t &= -1 \\ (g^{tt} u_t + g^{t\phi} u_\phi) u_t &= -1 \\ g^{tt} (u_t)^2 &= -1 \\ -\tfrac{1}{1-2M/r} (u_t)^2 &= -1 \\ -u_t = e &= \sqrt{1 - 2M/r}\ . \end{aligned} \tag{4}$$

Therefore, the specific energy is the same as that of a static particle in a Schwarzschild spacetime at r. Note that we used $u_\phi = 0$ to drop the $g^{t\phi} u_\phi$ term in the third step; one cannot do this in general, because (unlike in the Schwarzschild spacetime) $g^{t\phi} \neq 0$.

(b) The specific energy of a particle as measured in its own rest frame is *always* 1. That's practically the definition. Alternatively, in its own local rest frame there is no gravitational potential (spacetime is flat), and no velocity, so $E = mc^2$.

(c) The angular velocity as seen at infinity is again $\Omega = d\phi/dt$. Therefore,

$$\begin{aligned} \Omega = \tfrac{d\phi}{dt} &= u^\phi / u^t \\ &= g^{\phi\alpha} u_\alpha / (g^{t\beta} u_\beta) \\ &= (g^{\phi\phi} u_\phi + g^{\phi t} u_t) / (g^{tt} u_t + g^{t\phi} u_\phi) \\ &= g^{\phi t} u_t / (g^{tt} u_t) \\ &= g^{\phi t} / g^{tt} \\ &= 2 j M^2 / r^3\ . \end{aligned} \tag{5}$$

This shows that in this spacetime a particle with zero angular momentum does *not* have zero angular velocity as seen at infinity. That's the result of spacetime being dragged.