Any questions on homework?

High densities and strong magnetic fields

Both these topics have major importance in determining the energy transport properties of neutron stars, which is why I'm including it here.

Special topic: high densities

This will be a qualitative and descriptive discussion rather than a quantitative one, because the calculation of superfluid and superconducting gap energies in very dense matter is highly imprecise.

General idea: if there is an attractive pairing interaction between fermions, they can couple to form a state with integral spin, and therefore can act like bosons. At a low enough temperature, these "bosons" can form a condensate-like state in which all of the bosons occupy the same quantum state and form a superfluid. If the component fermions are charged, this forms a superconductor. In normal laboratory experience, the pairing is electronic and happens only at very low temperatures (other than the ceramic high T_c superconductors, which do their thing at liquid nitrogen temperatures or a bit above, almost all laboratory superfluid or superconducting phenomena are observed at temperatures less than 20 K). However, in the dense core of neutron stars, nucleonic pairing can happen.

As with all highly degenerate systems, pairing occurs between states near the Fermi surface (recall that in the cores of NS, both protons and neutrons are degenerate, just not relativistically so). Since there are many more neutrons than protons, neutrons and protons can't pair up easily because their momenta are substantially different. So, consider only $n-n$ and $p-p$ pairings. The first gives a superfluid, the second a superconductor.

Another general principle of phase transitions is that they happen only when it is energetically favorable. In this case, need to compare the pairing energy Δ with **Ask class:** what other energy? The thermal energy, kT. The pairing energy is *extremely* difficult to calculate from first principles; one reason is that the medium in which the pairings takes place makes a difference (particularly for the outnumbered protons). The value, however, is somewhere around 1 MeV. **Ask class:** what does that mean for the approximate transition temperatures? Since 1 eV is about 10^4 K, the transition temperature is around 10^{10} K (these are the *real* high-temperature superconductors of the universe!). The interior temperatures of neutron stars are expected to be less than this for stars older than a few hundred years at the most, so most of the mass of neutron stars is superconducting and superfluid(!). In the inner crust, between neutron drip (at about 4×10^{11} g cm^{-3}) and nuclear density (about 2×10^{14} g cm^{-3}), the free neutrons are probably paired in the $^1S_0$ state, as in BCS superconductivity (that is, their spins are opposite). At higher densities, calculations suggest they are in the $^3P_2$ state, with aligned spins. Protons are probably coupled in the

$^1S_0$ state to form a superconductor.

Ask class: since most of the mass of a neutron star is a superconducting superfluid, what consequences would this have for some of the bulk properties of the star, such as its thermal, rotational, or magnetic properties? A superfluid is irrotational, so any rotation must be quantized in vortices of ordinary matter. However, these vortices are close enough together (about 10^{-2} cm for a 30 Hz rotator like the Crab) that for many purposes one can treat the interior as rigidly rotating. A superfluid also is a perfect thermal conductor, so to an even greater extent than for normal degenerate matter, the interior of a neutron star is isothermal. A superconductor excludes magnetic flux, so any magnetic field in the interior must likewise be quantized in vortices.

Effects linked to superconducting superfluidity

Glitches.—As with many other things related to neutron stars, some of the cool physics can't be observed directly, but its consequences might be observed. One example is glitches. The typical rotational behavior of an isolated neutron star is to spin down slowly, as seen by radio pulses. However, on occasion some pulsars (e.g., the Crab or Vela pulsars) show sharp increases in their rotational frequencies (by 10^{-9} to 10^{-6}, fractionally). This happens on times scales less than minutes, and is called a "glitch". When first one was seen from Vela in 1969, some models proposed this would be a one-time event (Goldreich-Pines). Now thought that this is a result of sudden coupling between superfluid and normal matter in the crust. Only crust is magnetically braked, but when there is a critical $\Delta\Omega$ reached then there is a catastrophic unpinning of the vortices that couple the normal crust to the superfluid.

Magnetic field decay in NS.—This is a story of how theory and observation follow each other like the blind leading the blind! **Emphasize importance of understanding how inferred quantities are derived**. In 1969, Gunn and Ostriker plotted the inferred magnetic fields and ages of the first $\sim$30 pulsars, and found that the older NS appeared to have weaker magnetic fields, with a decay time of a few million years. Lots of theory followed. Then in 1975, Lyne et al. pointed out that this was an artifact of the way things were plotted: inferred magnetic field is $B \sim \sqrt{P\dot{P}}$, whereas inferred age is $T \sim P/\dot{P}$. The period P varies little over the sample of pulsars then known, but the period derivative $\dot{P}$ varies a lot, so effectively plotting B versus T was like plotting $\dot{P}^{1/2}$ versus $1/\dot{P}$, which guarantees the relation that was observed! Lyne et al. showed this by *randomly* reassigning the observed periods and period derivatives to each other and getting the same relation. Nonetheless, Lyne et al., in the *same* paper, came up with what they thought was an independent demonstration of the same decay effect, with about the same time scale. Now, it isn't believed, but is instead thought that neutron stars that are spun up by accretion somehow have their field decay.

Anyway, getting back to superstuff, one proposed idea for how this happens is that since in the superfluid superconductor both magnetic field and rotation are quantized in vorticies of normal fluid, what happens is that (1) accretion onto a strongly magnetized star slows it down, forcing rotational vortices outward, (2) these rotational vortices bring magnetic flux vortices with them, pushing the currents into ordinary matter with finite resistivity, (3) time passes, and the currents and field in the crust decay, (4) the star is now spun up by accretion, leaving a fast-rotating but low-field neutron star. Maybe 'tis, maybe 'tisn't. There is as yet no consensus.

Special topic: strong magnetic fields

This allows us to segue into strong magnetic fields.

As we have discussed, magnetic fields tend to have a minor impact on the structure of ordinary stars, and even a minor impact on their spectra (fields less than 10^5 G or so are difficult to detect in stars other than the Sun). For neutron stars, however, their effect on the spectrum is dominant, as is their effect on radiation transport properties, which are in turn the most important energy transport mechanisms near the surface of the star.

Let's start the treatment by thinking of a free electron spiraling in a magnetic field. Suppose that it is moving at an angular frequency ω in a circle of radius r, and apply the classical force balance equation. **Ask class:** what is the centrifugal force for an electron, of mass m_e? It's just $m_e\omega^2 r$. **Ask class:** if it is moving at an azimuthal velocity v_ϕ, what is the magnetic force on the electron for a magnetic field B? Simply eBv_ϕ/c. Equating the two, and using $v_\phi = \omega r$, we find the frequency (the cyclotron frequency)

$$\omega_c = \frac{eB}{m_e c} = 11.5\,\mathrm{keV}\,B_{12}\ , \tag{1}$$

where as usual the convention is that $B_{12} = B/10^{12}$ G. Another important quantity is the scale length of the radius of the orbit, which is given by the Bohr-Sommerfeld quantization rule:

$$L_z = m_e\omega r^2 \sim m\hbar \tag{2}$$

where $m = 0, \pm1, \pm2, \ldots$ is the azimuthal quantum number. Therefore, the radius of the mth orbital is something like

$$r_m \sim \hat{r}\sqrt{|m|}\ , \tag{3}$$

where $\hat{r} = (\hbar c/eB)^{1/2} = 2.5\times10^{-10}\,B_{12}^{-1/2}$ cm is the Landau radius. More exactly, $r_m = \hat{r}\sqrt{2|m|+1}$.

Ask class: we now want to estimate when the orbitals of atoms are affected significantly by the magnetic field. What should we compare? We could compare energies (cyclotron versus Coulomb), forces (magnetic versus Coulomb), or distances (size of atomic orbital versus radius of mth Landau orbital). All three give approximately the same answer. Using forces, equality happens when

$$\frac{Ze^2}{r_m^2} = \frac{ev_m B}{c} = \frac{e\omega_c r_m B}{c}\ , \tag{4}$$

giving the critical field for state m of a hydrogenic atom of nuclear charge Z as

$$B_c = \frac{Z^2}{(2m+1)^3} B_0\ . \tag{5}$$

The critical field for the ground state of hydrogen is $B_0 = m_e^2 ce^3/\hbar^3 = 2.35 \times 10^9$ G, at which point the cyclotron energy is 2×13.6 eV. **Ask class:** given this, are magnetic effects likely to be most important for ground states or excited states? For hydrogen or for heavier atoms? Excited states and low-charge nuclei are easiest to get in the magnetically dominated regime. In fact, lab experiments with high Rydberg level hydrogen ($n \sim 100 - 200!$) have shown some of these effects.

Ask class: consider now a neutron star with a magnetic field $\sim 10^{12}$ G. The energy difference between the ground state and first excited Landau state is $\hbar\omega_c$. At what approximate surface temperature would one expect excited Landau states to exist? At about $kT = 10$ keV, or $T \sim 10^8$ K. This is much hotter than most neutron star surfaces, which have $T \sim 10^5 - 10^6$ K.

When $B \gg B_c$ and $kT \ll \hbar\omega_c$, atoms become essentially cylindrical (simple when B is very small or very large, but complicated when B is intermediate). In this limit, the Coulomb force dominates only along the magnetic field, and the orbitals are tightly bound across the field. The length across the field for the ground state is therefore $\hat{r}$, and the length along the field is greater, with some length ℓ to be determined. The energy of these orbitals is

$$E \sim \frac{\hbar^2}{2m_e\ell^2} - \frac{Ze^2}{\ell} \ln(\ell/\hat{r})\ . \tag{6}$$

Minimizing with respect to ℓ gives

$$\ell \sim \left(\frac{a_0/Z\hat{r}}{\ln(a_0/Z\hat{r})}\right)\hat{r} \tag{7}$$

where $a_0 = 5 \times 10^{-9}$ cm is the Bohr radius. For example, when $B = 10^{12}$ G, $a_0 \approx 20\hat{r}$, so $\ell \approx 7\hat{r}$ for hydrogen. The energy is then

$$E \sim Z^2 \frac{\hbar^2}{m_e a_0^2} \ln^2\left(a_0/Z\hat{r}\right)\ . \tag{8}$$

The factor before the log is just the usual ground state energy without any magnetic field. For $B = 10^{12}$ G, the multiplying factor in this formula is about 9, predicting about 120 eV for the ground state energy of hydrogen. The real value is about 160 eV.

This is amazing! It means that for a typical NS magnetic field, the ground state energy is *ten times* what it is for no field! This makes a huge difference in many ways. For example, remember our friend the Saha equation? It has to be modified in strong fields in other ways, but if we applied it blindly for no magnetic field in a neutron star atmosphere we'd find it predicted no neutron hydrogen. But the binding energy is so huge that hydrogen can exist at even $\sim 10^6$ K in the neutral state. This means that there is a significant extra source of opacity here. In addition, it turns out that the strong-field equivalent of the Kramers type opacity is that the frequency dependence of bound-free or free-free is not like ν^{-3}, but like ν^{-1}, so it drops off more slowly.

Given this, class is almost ready to come up with one of the major results of my thesis. Cooling of neutron stars is interesting in many ways, for example it gives hints about the internal structure of these stars (e.g., the presence of weird stuff at high densities, or of superfluidity, could make a big difference). However, we can't see all the emission from NS; cooling emission is predominantly in the X-rays, and X-ray satellites have a limited range of energies that they can observe. Analyses prior to mine had only considered nonmagnetic atmospheres. Given that these atmospheres are dominated by nonmagnetic Kramers opacities, **Ask class:** would we expect higher or lower photon energies to come through more easily? Higher energies, because of ν^{-3} effect. This means that in these bands, there is lots more emission than their would be from a blackbody, given the same effective temperature. This would mean that NS are a lot cooler than blackbody fits would indicate, and it's in that direction that the neat stuff (quarks, hyperons, superfluidity) would operate.

However, **Ask class:** what happens when the field is really strong, given that bound-free and free-free are still important? The weaker frequency dependence, plus the fact that hydrogen and helium transition energies are closer to the window, means that the opacity is a lot flatter, and as a result blackbody fits aren't as bad as people thought, and there is no evidence for rapid cooling of NS. With higher-resolution instruments such as Chandra and XMM, might find high-field line transitions that could confirm a lot of my work. I've been waiting a long time!

Ultrastrong fields and magnetars

Finally, let's think about some other strange effects that happen when the field gets really strong. In general, photons propagating in a magnetic field at some direction to the field have two normal modes of polarization. The "ordinary" mode has a polarization along the field, and the "extraordinary" mode has a polarization across the field (generically speaking). A scattering, like Thomson scattering, involves shaking an electron up and down in the direction of the polarization. It should therefore be no surprise that the cross section to ordinary mode scattering is a lot large than to extraordinary mode, because in the ordinary mode the electron can move freely (along the field) whereas in the extraordinary mode it can't move so freely, since that would take it across the field. In fact, ignoring some angle-dependent factors, the scattering cross section in the ordinary mode is about σ_T, whereas across the field is about $\sigma_T(\omega/\omega_c)^2$, where ω is the frequency of the photon. For $B = 10^{12}$ G and a photon energy of 1 keV, **Ask class:** what would the approximate reduction of cross section be? About a factor of 100.

Ask class: we've had as a theme throughout the energy transfer part of this course that sometimes opacities add linearly, and sometimes they add harmonically. Which would it be in this case? Harmonically, because the two polarizations are different channels. That means that the energy flux is dominated by extraordinary mode photons. **Ask class:** related to their problem set, problem 1, what would this much lower cross section mean for the maximum luminosity from a strongly magnetic source? It would be higher. The factor by which it is higher has some subtleties, but it could be 10^4 times higher than usual for a "magnetar" in which the magnetic field is 10^{15} G or so.

The last thing to mention is that when you have high photon energies and high magnetic fields there are other exotic processes that can operate, and could even dominate for magnetars. One is photon splitting: $\gamma B \longrightarrow \gamma\gamma B$. This is forbidden in free space, but the magnetic field can absorb momentum and allow it to happen. Very high-order process, though. Another is magnetic one-photon pair production, which is thought to be essential in understanding why rotation-powered pulsars generate radio waves. In this process, $\gamma B \rightarrow e^- e^+ B$, a photon with an energy at least $2m_e c^2$ moves perpendicularly to the field and creates a pair. This is a process with an exponential onset: even if the energy criterion is met, it only happens for $\Gamma(B/B_{\rm QED})\sin\theta > 0.05$, where $B_{\rm QED} = 4.4 \times 10^{13}$ G is the field at which the electron cyclotron energy equals the electron rest mass. Above this threshold, the process happens so fast (centimeters) that it dominates other processes.

These processes, such as photon splitting, are interesting in a very fundamental way. Some of them can only happen in highly magnetized neutron stars, and can't be duplicated in a laboratory. That means that NS can be used as probes of a completely new physical regime.