The week before spring break (i.e., the week after the midterm), I'd like 15 minutes with each person individually to hear suggestions for their end of term project and about the class in general and how it could be improved.

The midterm will be Tuesday, March 12. Should it be open or closed book? I will be away from Saturday through Wednesday inclusive, so I'll be happy to help with midterm preparation until then.

This homework was tougher than the last, as I expected it would be. Common mistakes: in problem 1, even if an individual term is really small, if there are lots of them you still have to sum them to see if the sum is insignificant. In problems 3 and 4, remember that if something shrinks then it will heat up and if something expands it will cool. In problem 4, remember that the ideal gas pressure depends on the total number density, so if you have two ionized electrons per iron nucleus then the number density is three times the number density of the iron nuclei alone. Also in problem 4, remember that the mean molecular weight per free electron, μ_e, is the number of nucleons per free electron. For example, at an ionization fraction of 10^{-5} with iron, $\mu_e = 56/10^{-5} = 5.6 \times 10^6$.

Want to recap when different opacities are important, and the difference between cross section and opacity, so part at the end is important.

Gray atmosphere: Won't go into stellar atmospheres much in this course, but it is useful to know that for many purposes the gray atmosphere solution (using the Rosseland mean opacity) is remarkably good. In this solution, if τ_R represents the Rosseland optical depth from the surface, the temperature as a function of depth is given by

$$T^4 \approx \frac{3}{4} T_{\rm eff}^4 (\tau_R + 2/3) \ , \tag{1}$$

where $T_{\rm eff}$ is the effective temperature, such that the radiated flux from the surface is $\sigma T_{\rm eff}^4$. This isn't perfect, of course, but it's surprisingly accurate even when there are lines and edges and stuff. The more important difference from gray atmospheres that exists in real atmospheres is that the emergent spectrum is usually much different.

Radiative opacities

Radiative opacity sources: Ask class: what are the ways in which a photon can interact? Can be done off of free electrons, atoms, molecules, or dust. Can also interact with protons/nuclei, magnetic fields, and other photons, but in this class we'll ignore those for the most part. For free electrons, have electron scattering (not an absorption) and free-free (inverse bremsstrahlung, which is an absorption). Scattering means that the photon retains its identity (in a somewhat ill-determined way), whereas absorption means that the photon is destroyed and later another one is emitted.

For atoms, have transitions between atomic levels (bound-bound absorption) and ionizations (bound-free absorption). **Ask class:** when do they expect opacities involving free electrons to be most important? When ionization fraction is high. Molecules and dust are more important for lower-energy photons. Same with H^- opacity, which is important for optical opacity in some stars.

Scattering from free electrons: Let's start with Thomson scattering. Fairly straightforward, really: for photon energies much less than the mass-energy of an electron, the total cross section is a constant:

$$\sigma_T = 6.65 \times 10^{-25}\,\mathrm{cm}^2\ . \tag{2}$$

Scattering is not isotropic: the differential cross section for a direction θ away from the initial propagation direction of the photon is $d\sigma/d\Omega \sim (1+\cos^2\theta)$. Thus, photons undergoing Thomson scattering are not exactly doing a random walk, since they have a greater chance of going forward or backward than to the side. Makes virtually no difference to the final result, though. **Ask class**: think of Thomson scattering as the oscillating electric field of the photon causing the electron to wiggle back and forth, and hence radiating. Do they expect it to be easier or harder to do the same thing with a proton? Harder, because the proton has more inertia, so it accelerates less and therefore radiates less. So, **Ask class:** should the Thomson scattering cross section be more or less for protons? The Thomson cross section is proportional to the mass of the particle to the –2 power, so scattering from ions is almost always negligible.

Pure Thomson scattering is elastic, so photons retain their energy exactly. **Ask class:** is this true in reality? In particular, think of the process in the frame in which the electron was initially at rest. There must be some recoil, so the photon after scattering has on average changed its wavelength by of order the wavelength of a photon whose energy is the electron rest-mass energy (i.e., the electron Compton wavelength):

$$\lambda_f - \lambda_i = \lambda_c(1-\cos\theta) = \frac{h}{mc}(1-\cos\theta) \tag{3}$$

where λ_i is the initial, and λ_f the final, wavelength of the photon in the original electron rest frame. In the electron rest frame the photon always loses energy, but in the lab frame the electron, if moving fast enough, can increase the energy of the photon (called inverse Compton scattering). When both the electrons and photons are in isotropic thermal distributions, equilibrium is reached (not surprisingly) when the radiation and electron temperatures are equal.

The recoil effects also modify the scattering cross section somewhat and make it frequency-dependent. For initial photon energy ϵ_i, final photon energy ϵ_f, and a photon direction after scattering that is θ away from the initial direction, the differential cross section in the initial electron rest frame is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - \sin^2\theta\right) , \tag{4}$$

where $r_0 = e^2/m_e c^2$ is the classical radius of the electron. This is the Klein-Nishina formula, and it causes significant deviations only when the photon energy is comparable to the electron rest mass energy. If $x \equiv \hbar\omega/m_e c^2 \ll 1$, then the approximate total cross section is $\sigma \approx \sigma_T(1-2x)$. This is therefore only a big deal above about 100 keV. Scattering is often a minor player for photon energies below ∼keV or so, corresponding to $\sim 10^7$ K, because at lower energies absorption processes dominate.

One last bit of scattering to mention for completeness is cyclotron scattering. This occurs when the electron cyclotron energy $\hbar\omega_c = \hbar eB/m_e c \approx 10(B/10^{12}\,\mathrm{G})$ keV is in the frequency range of interest. The cross section for scattering is $\sim 10^6$ times the Thomson cross section, so in the right circumstances (the atmosphere of a neutron star or of a very magnetic white dwarf) it can have a major effect on the emitted spectrum, but for normal stars the energy is way below the energies of interest and it has a negligible effect.

Bound-bound absorption This is when a bound electron absorbs a photon and consequently makes a transition to a higher bound level. An electron in such a higher level has a finite lifetime to decay Δt. **Ask class:** what does this finite lifetime mean about the energy width Γ of the transition? From the uncertainty principle, $\Gamma = \hbar/\Delta t$. Applying time-dependent perturbation theory (see, e.g., Clayton), the cross section of a transition from initial state s to final state k as a function of energy can be written in the Lorentzian form

$$\sigma(\omega) = \frac{2\pi^2 e^2}{m_e c} f_{\rm ks} \frac{\Gamma/2\pi\hbar}{(\omega - \omega_{\rm ks})^2 + (\Gamma/2\hbar)^2} . \tag{5}$$

Here $f_{\rm ks}$ is the "oscillator strength". There's a sum rule saying that the sum of the oscillator strengths of all transitions of an electron from a given energy level is 1; caution, though, because emission counts negatively for the sum. Most oscillator strengths are less than or of order 1. Story about my $f = 10^6$!

As an example, take the Lyα resonance of hydrogen, where an electron in the 1s state is absorbed into the 2p state. This has an oscillator strength of 0.42 and a peak cross section of 7.2×10^{-11} cm. This is *enormous*, a good 10^{14} times the Thomson cross section. However, it is important over only a very small range in energy. The lifetime is about 10^{-9} s, so the energy width is $\hbar/10^{-9} \approx 10^{-6}$ eV.

Generally, therefore, bound-bound transitions have extremely high cross sections over extremely small regions of the spectrum. Given that the Rosseland mean is harmonically weighted, I would have expected that bound-bound transitions, although producing all of the lines that we see in spectra, would be unimportant in energy transport. People in the

1960s and earlier thought so, too, but it turns out that the total opacity in nonionized portions of the Sun can be increased by a factor of 2 or 3 by bound-bound transitions. This is because with all the transitions available from metals, the wings overlap.

Bound-free absorption

The process here is that a photon ejects an electron from a bound state inside an atom. **Ask class:** let's say you have a photon whose energy is much larger than this threshold energy. Do you expect the cross section (i.e., strength of interaction) to increase or decrease with increasing energy? Decrease, because it's closer to the resonance energy of ionization. The cross section falls off like ν^{-3} above a threshold. Below the threshold there is no absorption, so the total cross section looks like a sawtooth (draw). The cross section for hydrogenic atoms is

$$\sigma_{\rm b-f} \approx 2.8 \times 10^{29} \frac{Z^4}{n^5 \nu^3} g \ {\rm cm}^2 \ , \tag{6}$$

where $g = g(\nu, n, l, Z)$ is the Gaunt factor and is close to unity, n is the principal quantum number, Z is the charge of the nucleus, and ν is measured in Hertz. At the hydrogen edge this becomes close to 10^{-17} cm^2, which is much less than the peak of a bound-bound transition but much more than the Thomson cross section. Using the ν^{-3} shape, bound-free becomes competitive with Thomson scattering for half-ionized hydrogen when the photon energy is about 3 keV.

To get the opacity requires multiplying this by the number of ions in this state per gram. Hansen & Kawaler give the expression

$$\kappa_{\rm b-f} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} \,{\rm cm}^2\,{\rm g}^{-1} \ , \tag{7}$$

where Z is now the fractional abundance of metals. **Ask class:** what's wrong with this expression, or more correctly, what is the domain of applicability? This expression would predict zero bound-free absorption in pure hydrogen, if used blindly. The real point is that the elements with ionization energies comparable to the temperature will contribute most, and at most temperatures of interest in stars the hydrogen is fully ionized. Therefore, as the frequency of the photons and the temperature go up, one metal/atomic state takes over where another has left off. The net result in the interstellar medium is that at far UV to X-ray energies the opacity is dominated by bound-free absorption, and actually ends up having the approximate form $\kappa \sim \nu^{-8/3}$ because of the relative abundances of elements. This means that at low energies (below around 0.5 keV or so, depending on the column depth) the spectrum appears cut off to us.

This expression for the opacity, with a factor $\rho T^{-3.5}$, is called a Kramers opacity. It also applies to free-free absorption, as we will see, and is a very generally useful form of the opacity for much of the interior of stars.

Free-free absorption

In this process a free electron passes by an ion of charge Z, scatters off of it, and absorbs a photon. This is the inverse process to bremsstrahlung. Conservation of energy means that

$$\frac{p_k^2}{2m_e} = \frac{p_s^2}{2m_e} + \hbar\omega \ . \tag{8}$$

Here, of course, we're assuming nonrelativistic electrons. The presence of an ion is very important. For example, suppose we just want an electron to emit a photon. Then it is impossible to conserve both energy and momentum.

The fundamental reason for bremsstrahlung, or braking radiation, is that as an electron moves past an ion it is accelerated and therefore radiates power:

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}a^2(t) \ , \tag{9}$$

where $a(t)$ is the time-dependent acceleration.

The cross section for free-free absorption around an ion of charge Z is

$$\sigma_{\mathbf{f-f}} = 3.7 \times 10^8 \frac{Z^2 n_e g_{\mathrm{f-f}}}{T^{1/2}\nu^3} \ \mathrm{cm}^2 \ . \tag{10}$$

In a thermal distribution, we can take the Rosseland mean opacity. For a temperature T, the peak in the blackbody distribution is around $3kT$. **Ask class:** would you expect the Rosseland weighting to give an effective ("weighted average") photon energy greater than or less than this? Greater than, because the Rosseland mean is a *harmonic* mean, implying that lower opacities will get higher weight. Since the cross section drops with increasing photon energy, higher energies will get greater weight and the effective mean photon energy is greater than the energy of the blackbody peak (in fact, about $\langle h\nu\rangle_{\mathrm{eff}} \approx 5.8kT$).

A reasonable approximation for the free-free opacity is

$$\kappa_{\mathrm{f-f}} \approx 10^{23}\frac{\rho}{\mu_e}\frac{Z_c^2}{\mu_I}T^{-3.5} \ \mathrm{cm}^2\ \mathrm{g}^{-1} \ , \tag{11}$$

where μ_e and μ_I are the mean molecular weights that we've used before, *but* remember that only free electrons contribute!

The Total Radiative Opacity

At this point, let's step back and think about what this means. In principle, a photon propagating through gas will have some probability of interacting in all of the ways above (plus others we haven't discussed as much, like bound-bound absorption, pair production for high energy photons, and so on). We need a way to determine what dominates when.

Ask class: Suppose we suppress all photon interaction mechanisms except for one (scattering for definiteness). Let's consider photons of a given energy. What factors determine the mean free path of those photons? The cross section and the number density of scatterers. Specifically, $\ell = 1/n\sigma$. **Ask class:** is n the number density of gas particles in any state, or just those that can scatter? Only those that can scatter. **Ask class:** so, suppose we have a gas of 100% neutral hydrogen atoms. What is the mean free path to Thomson scattering? It's infinite, because there are no free electrons from which the photons would scatter.

This points out a difference between *cross section* and *opacity*. The cross section says (in this case): suppose you have free electrons. What is the surface density (in cm^{-2}) of those free electrons so that only a fraction e^{-1} of photons go unscattered? That's just $\Sigma = 1/\sigma$. But this does *not* tell you how important scattering is! For that, you need to know the number density of the scatterers. Opacity is expressed in terms of $\kappa =$cm^2 g^{-1}. How many scatterers (each with cross section σ) are there per gram of material? κ/σ.

So, **Ask class:** suppose you have hydrogen gas that is 100% ionized. How important do you expect bound-free ionization to be? Totally unimportant, because there are no bound atoms! How about bound-bound? Same thing. Of the processes we've considered, only free-free and scattering could possibly contribute.

Okay, now let's suppose we have photons of a given energy. We've computed the opacities for all the different processes. **Ask class:** which processes will dominate, those with high opacities or those with low opacities? High opacities, since those interactions will occur first. At a given frequency, high opacities win.

Now, suppose that we have a flat input spectrum of photon energies. We send it into a region of gas, where the total opacity varies as a function of energy. **Ask class:** do they expect most of the radiation transferred through this region to be at frequencies of high opacity or low opacity? Low opacity, because it is easier to propagate photons that way. That's why in optically thick plasmas (like the Sun!), lines tend to be dark, because there's less emission there. In fact, when one considers the "effective" opacity over a spectrum in an optically thick region, one usually takes a harmonic average (that is, weighted by the reciprocal of the opacity). This is called a Rosseland mean.

Summary: opacity, not cross section, determines importance of processes. At a given photon energy, high opacities dominate. Over a spectrum, low opacities dominate.