Equations of state: interactions and high density

Coulomb interactions: Here's where we get to think about some of the effects producing states of matter (gas, liquid, solid) in ordinary matter. These nonideal effects come in when electrostatic interactions between atoms or molecules become significant. **Ask class**: if we have defined some characteristic electrostatic energy E_c, what is a reasonable energy to compare it to in order to determine whether it is an important correction? (thermal energy kT or Fermi energy E_F). If we assume that the gas is fully ionized and therefore consists, to a first approximation, of a uniform distribution of electrons plus positive charges concentrated in individual nuclei of charge Z, then the electrostatic energy is approximately $E_c = Ze^2/\langle r\rangle = Ze^2 n_e^{1/3}$. Therefore, the ratio of Coulomb (electrostatic) energy to thermal energy is

$$\frac{E_c}{kT} = \frac{Ze^2 n_e^{1/3}}{kT} \tag{1}$$

and the ratio of Coulomb to Fermi energy (nonrelativistic) is

$$\frac{E_c}{E_F} = \frac{Ze^2 n_e^{1/3}}{p_F^2/2m_e} = \left(\frac{n_e}{Z^3 \times 6 \times 10^{22}\ \mathrm{cm}^{-3}}\right)^{-1/3} . \tag{2}$$

Not a major correction for most degenerate gases.

Wigner-Seitz approximation: suppose we divide things up into spherical "cells" of volume $1/n_N = 4\pi r_0^3/3$ and assume that these are neutral, containing the nucleus of charge Z and a uniform sphere of electrons of total charge $-Z$. From the book and notes, the total electrostatic energy per electron is then the electrostatic energy of the electrons with themselves, plus the energy of the electrons with the protons. The total is

$$E_c/Z = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3} . \tag{3}$$

Ask class: how can we tell if this is reasonable? For example, is it correct that the energy is negative? Yes, because this configuration is bound. Is it correct that the energy depends directly on density? Yes, more tightly squeezed means greater electrostatic binding energy.

The notes give the pressure correction due to Coulomb interactions in this model, for relativistic and nonrelativistic limits. In the nonrelativistic limit,

$$P/P_0 = 1 - \frac{Z^{2/3}}{2^{1/3}\pi a_0 n_e^{1/3}} , \tag{4}$$

where $a_0 = \hbar^2/me^2 \approx 5 \times 10^{-9}$ cm is the Bohr radius. Thus, these are small corrections for high densities. **Ask class:** does this equation make sense at all nonrelativistic densities? No! In fact, one could take $n_e \to 0$ and find that $P/P_0 \to -\infty$. Not good. Another problem is apparent if you ask when $P = 0$, which would be the equilibrium state of zero-pressure material. This formula would predict that the pressure is zero (implying stable matter) at $\rho_0 = 0.4Z^2$ g cm^{-3}, or 250 g cm^{-3} for iron instead of 7.86! **Ask class:** recalling that we assumed uniformly spread negative charges, can they think of a way in which the Wigner-Seitz approximation may fail at low densities? Reason is that for low densities, the electrons cluster much more towards the nuclei. Accounting for such things is the basis of Thomas-Fermi or Thomas-Fermi-Dirac models, but that would take us too far afield.

Formation of crystalline lattice: **Ask class:** what things would one expect to compare when trying to determine whether a crystalline lattice will form? It is again an energetics thing; when the lattice energy is large enough compared to the temperature and Fermi energy of the *ions* (not talking about electrons at this point), can go from a liquid phase to a crystalline, solid phase in which the Coulomb energy is minimized. **Ask class:** When we're talking about ions, would one expect the thermal energy or Fermi energy to dominate? Usually thermal, because the Fermi energy is a factor m_e/m_i lower than for electrons, and for most densities is small. Therefore, want to compare $(Ze)^2/r_i$, the Coulomb energy, with kT, the thermal energy. The full calculation is complicated, but one can eventually get an estimate of the critical ratio using Lindemann's empirical rule that the lattice will melt when the mean square fluctuations in the ion position become

$$\frac{\langle(\delta r_i)^2\rangle}{r_i^2} \sim \frac{1}{16} \tag{5}$$

or more. Full inclusion of lots of stuff gives that the critical ratio Γ between the Coulomb energy and the thermal energy necessary to crystallize is $\Gamma \approx 170$. This is important in the late stages of white dwarf cooling, and provides an extra source of energy.

Ask class: What does this estimate mean for iron? $Z = 26$, typical density of solid iron is about 8 g cm^{-3} and $A = 56$, so $n \approx 8/(56 \times 1.7 \times 10^{-24}) \approx 8 \times 10^{22}$ cm^{-3}, or about 10^{-23} cm^3 per atom. That's about $r = 1.4 \times 10^{-8}$ cm. So, $(Ze)^2/r \approx 10^{-8}$ erg per atom. Dividing this by 170, we get 6×10^{-11} erg, or about 500,000 K! Wow! **Ask class**: what went wrong? Answer: we used the full, bare charge of the iron nucleus. In reality, the electrons in inner orbitals shield the iron, so the effective charge is much less. Doesn't apply when the electrons are uniformly spread. Therefore, the Coulomb energy is a lot less as well, and the melting temperature is more like 2,000 K.

Magnetic fields: As an aside, magnetic fields can affect the equation of state or properties of the matter in two ways. One is to make a contribution to the pressure: **Ask class**: what is the magnetic pressure? $P_B = B^2/8\pi$; for example, in the center of the Sun, the ideal gas pressure nkT is about 10^{17} cgs, requiring $B \sim 10^9$ G to compete. The other is to change the opacities significantly. Zeeman splitting can be seen in laboratories even with extremely weak fields, but to really make a big difference, need the cyclotron energy $\hbar\omega_c = \hbar eB/mc$ to be comparable to the binding energy of an electron. This means

$B \sim 10^9$ G again for hydrogen, which is only true in neutron stars or possibly some white dwarfs. Note that for more excited states, in which the atomic binding energy is reduced, threshold magnetic field is less.

Higher densities: Let us now think about what happens to the composition of the matter as the density increases. Consider a fully ionized gas of protons, neutrons, and electrons (i.e., no more complicated nuclei). **Ask class:** at low density, what is the equilibrium composition? Just protons and electrons, because neutrons are unstable. That's because the neutron rest mass is greater than the sum of the rest masses of the proton and electron. **Ask class:** energetically, when would one expect it to be favorable for neutrons to exist? When the total energy of the electron becomes large enough that $m_n c^2 \leq m_p c^2 + E_e$, neutrons will be favored. This will occur when the Fermi energy of the electrons plus their rest mass is large enough; $c^2(m_n - m_p - m_e) \approx 1.2$ MeV, so that's what E_F needs to be. **Ask class:** given that electrons, with a mass-energy of 511 keV, become relativistically degenerate at $\rho \sim 10^6$ g cm^{-3}, at approximately what density would we expect their Fermi energy to be 1.2 MeV? In relativistic regime, $E_F \sim p_F \sim n^{1/3}$, so density is about $\rho \approx 10^6(1.2/0.511)^3 \approx 10^7$ g cm^{-3}.

Important point: When there is a density-induced phase transition like this, in which it is energetically favorable to change composition, the equation of state becomes "soft", meaning that the energy density (and hence the pressure) does not increase as rapidly with increasing density as it had been, so it's easier to squeeze. This has crucial consequences for a number of systems. For example, early universe QCD phase transition; when matter when from being a quark-gluon soup to being nuclei, the universe was extra squeezeable and black holes may have formed directly. In principle this could mean that black holes of a few to a few hundred solar masses could comprise dark matter in the universe; their formation occured before big bang nucleosynthesis, so matter in this form would not interfere with BBN. On the other hand, constraints are strong on black holes as dark matter. Density-induced phase transitions may also take place in the cores of neutron stars. Note that this means there are places in a white dwarf or neutron star where, conceivably, the gamma-law index could be well below 4/3. The star as a whole is not necessarily unstable, though, because most of it is in the stable regime so large-scale collapse would increase the pressure dramatically.

On to neutron drip: (Qualitative only). In reality, there are nuclei around instead of just pne. At low densities, equilibrium nucleus is ^{56}Fe. **Ask class:** do they remember what I said about this being qualitatively explained by a competition between nuclear and electrostatic forces? Nuclear is much stronger but short-range, so more nucleons increase the binding energy. However, Coulomb energy decreases the binding energy because of repulsion between protons. As the nucleus grows in size the Coulomb energy grows in importance relative to the nuclear energy, and a balance is struck in equilibrium with iron.

Ask class: since nuclear energy is short-range, why couldn't we just make large nuclei a lot denser? Because Fermi energy would grow in compensation.

Anyway, as the density increases, the increased Fermi energy of the electrons means that, overall, it becomes energetically more favorable to have extra neutrons, so the equilibrium nuclei become progressively more neutron-rich. The calculation of the equilibrium nucleus at a given density is complicated, including things like projections of laboratory mass formulas and lattice effects; revisions have happened in the last decade. At about $\rho = 4\times 10^{11}$ g cm^{-3}, when the equilibrium nuclei are things like ^{118}Kr, the outermost neutrons in the nucleus have zero total energy and hence can "drip" out of the nucleus, producing a system of electrons, neutrons, and nuclei.

Mention: Although it is possible to calculate the equilibrium nucleus at a given density there is no guarantee that everything at that density will be in that state! Good things for us, too, otherwise we'd all be lumps of iron. The same is true at high densities, and it can have important effects in certain circumstances. For example, impurities can change the conductivity in white dwarfs or neutron stars substantially.

Higher densities: Above neutron drip, as density increases the composition becomes more and more neutron-dominated, with some nuclei in a sea of neutrons. When care is taken to determine the energy of nuclei, including surface and volume terms, it is found that as the density increases the shapes of the nuclei change. (Remember that nuclear density is 2.7×10^{14} g cm^{-3}, so at neutron drip we're still a factor of 1000 away). First, spherical nuclei in a tenuous(!) sea. Then elongated line-like nuclei. Then 2-D sheets with gaps in between. Then the gaps shrink, then space is mostly nucleons with line-like "holes", then mostly nucleons with spherical "holes". Dubbed the "pasta-antipasta" sequence: meatballs, spaghetti, lasagna, anti-lasagna, anti-spaghetti, anti-meatballs (Swiss cheese?).

At about nuclear density, mostly ($\sim$90%) neutrons plus equal numbers of protons and electrons. Might have other components at higher energies, but not clear. As an example, **Ask class** at what densities we might see things like lambda hyperons, which have a rest mass of 1116 MeV? Roughly expect this when the Fermi energy of the neutrons plus the neutron rest mass is around 1116 MeV, at which point it can be energetically favorable to produce λ instead (there are complicating issues such as the interaction energy between the Λ and the nucleons, but ignore those). At nuclear density, $E_F(n)\approx 60$ MeV, need it to be about 160 MeV because $m_nc^2\approx 940$ MeV. **Ask class:** how do we scale this? Non-relativistic, so $E_F\sim p_F^2\sim n^{2/3}$. Therefore, need density to be about $(160/60)^{3/2}$ times nuclear, or around 10^{15} g cm^{-3}. Pretty high, but it might be reachable. Any such new components (strange matter, quark matter, hyperons, kaons, pions, etc.) tend to soften the equation of state, which would mean that the maximum mass is less.