Any questions about homework?

Equations of state: degeneracy and interactions

Fermi-Dirac particles: Suppose we have a particle such as an electron, proton, or neutron that is spin 1/2 and therefore a fermion. Let the energy reference level be mc^2 (again, other choices are possible!). Then the degeneracy is 2, so the number density is

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{[-\mu+mc^2+E(p)]/kT}+1}\ . \tag{1}$$

In general, **Ask class**: $E(p) = mc^2\left[\sqrt{1+(p/mc)^2}-1\right]$ and the velocity is

$$v(p) = \frac{\partial E}{\partial p} = \frac{p}{m}\left[1+\left(\frac{p}{mc}\right)^2\right]^{-1/2}\ . \tag{2}$$

Ask class: how can we test whether this equation is correct? Limits for $p \to 0$, where $v = p/m$ (correct) and $p \to \infty$, where $v = c$ (correct). How about parity? v is in same direction as p, which is right. **Ask class:** from this, what is a rough boundary in momentum between relativistic and non-relativistic? When $p \approx mc$.

Completely degenerate gas: what is degeneracy? **Ask class**: it's when the density is high enough that particles start encroaching on each other's states. **Ask class:** what are the different implications for fermions and for bosons? For fermions this leads to Pauli exclusion and the fermions are forced into other energy states, whereas with bosons this leads to multiple occupation of the same state. The result is a host of phenomena, such as lasers, Bose condensation, superfluidity, and superconductivity. Some interesting properties can be derived from the fact that the particles participating in these phenomena are in a single state. For example, superfluids can't rotate because the particles would be moving with different velocities and hence would not be in the same state. Instead, any rotation in a superfluid is quantized in vortices of normal fluid.

Anyway, complete degeneracy occurs when $kT \to 0$, or more precisely when kT is much less than the numerator of the exponent above. The interesting part of the integrand is

$$F(E) = \frac{1}{e^{[E-(\mu-mc^2)]/kT]}+1}\ . \tag{3}$$

Ask class: what are the values for $kT \to 0$ when the exponent is positive and when it is negative? Clearly, either 0 or 1 for $kT \to 0$, depending on whether, respectively, $E > (\mu - mc^2)$ or $E < (\mu - mc^2)$. In this case, therefore, there is complete occupation up to the "Fermi energy" $E_F = \mu - mc^2$ and no occupation beyond that. When kT is finite, the transition between occupation and no occupation occupies a width $\sim kT$ in energy.

Only these particles can interact, a fact which has great importance for things like energy transfer. Specifically, it means that only a fraction $\sim (kT/E_F)$ of particles can interact, so mean free paths are a lot longer than one would imagine and in this limit conduction can be very important. **Note:** presaging our discussion of opacities, energy transfer tends to be dominated by whatever process can transport energy over a long distance rapidly. If there is a lot of scattering/absorption, this slows down the carrier particles. In degenerate matter, electrons travel a long way and hence conduction can be important; similarly, in a metal the periodicity of the potential cuts down on interactions and allows the electrons to travel far.

Ask class: do they remember how to get the Fermi momentum from the uncertainty principle? $\Delta p \Delta x > \hbar$, so for a number density n we have $p_F \sim \hbar n^{1/3}$.

The "Fermi momentum" is the momentum corresponding to the Fermi energy, and is designated p_F. A dimensionless parameter $x = x_F \equiv p_F/mc$ is often used. Therefore,

$$E_F = mc^2 \left[(1 + x_F^2)^{1/2} - 1 \right] \ . \tag{4}$$

Therefore, the chemical potential is $\mu_F = E_F + mc^2$, which is the total energy of the most energetic particles in the system. That is, it's the "energetic cost" of adding another particle of this type.

Ask class: given this $kT \to 0$ approximation and the consequences for occupation number, how does the integral for the number density simplify? Integrate up to p_F, with an integrand of $F(E) = 1$, then stop because there is no more occupation. Therefore,

$$n = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, dp = \frac{8\pi}{3} \left(\frac{h}{mc} \right)^{-3} x_F^3 \ . \tag{5}$$

Numerically, and dropping the "F" subscript,

$$n = 5.9 \times 10^{29} \left(\frac{m}{m_e} \right)^3 x^3 \, \mathrm{cm}^{-3} \ . \tag{6}$$

In density units, you have $\rho/\mu_e \approx 10^6 x^3$ g cm^{-3} for electrons (usually $\mu_e \approx 2$ in WD). The line between nonrelativistic and relativistic is **Ask class**: approximately $x = 1$ (as can be seen from the $x = p/mc$ definition), which is $\sim 10^6$ g cm^{-3} for electrons. This is a useful number to remember: when the density exceeds 10^6, the electrons are relativistic. **Ask class:** when the electrons are nonrelativistic, how does the Fermi energy scale with density? Let's work this through. **Ask class:** when the electrons are nonrelativistic, how does their Fermi energy depend on the Fermi momentum? Like $E \sim p^2$. **Ask class:** whether or not the electrons are relativistic, how does the Fermi momentum depend on density? Like $p \sim \rho^{1/3}$. Therefore, $E_F \sim \rho^{2/3}$ in the nonrelativistic limit. **Ask class:** how does it scale in the relativistic limit? By same reasoning, like $E_F \sim \rho^{1/3}$. Okay, so we'd like an idea of when the Fermi energy is "important". That means that we should compare it with other relevant energies. **Ask class:** what are some other relevant energies? Thermal energy and Coulomb energy come to mind. This will help them get a handle on problem 4. Note that $m_e c^2$ is about 511 keV, which corresponds to about 6×10^9 K.

For neutrons the density is 6×10^{15} g cm^{-3}, which substantially exceeds nuclear density of 2.7×10^{14} and is a few times larger than is expected to exist even in the cores of neutron stars, so neutrons are mostly nonrelativistic.

In a similar way, the pressure and internal energy density can be computed by integrating up to the sharp cutoff p_F. We find:

$$P_e = \frac{\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} m_e c^2 f(x) \equiv A f(x) \ , \tag{7}$$

where $f(x) = x(2x^2 - 3)(1 + x^2)^{1/2} + 3\sinh^{-1} x$, and

$$E_e = A g(x) \ , \tag{8}$$

where $g(x) = 8x^3\left[(1 + x^2)^{1/2} - 1\right] - f(x)$. Limiting forms are:

$$\begin{aligned} f(x) &\longrightarrow \tfrac{8}{5}x^5 - \tfrac{4}{7}x^7 + \cdots \ , \qquad & x \ll 1 \\ &\longrightarrow 2x^4 - 2x^2 + \cdots \ , \qquad & x \gg 1 \end{aligned} \tag{9}$$

and

$$\begin{aligned} g(x) &\longrightarrow \tfrac{12}{5}x^5 - \tfrac{3}{7}x^7 + \cdots \ , \qquad & x \ll 1 \\ &\longrightarrow 6x^4 - 8x^3 + \cdots \ , \qquad & x \gg 1 \end{aligned} \tag{10}$$

Ask class: taking only the leading order terms, and comparing the expressions for the pressure and energy density with each other, what is the γ in a gamma-law equation of state for nonrelativistic ($x \ll 1$) and relativistic ($x \gg 1$) degeneracy? 5/3 and 4/3, respectively. Note that because $n \propto x^3$ for any x, you can also write this in the form $P \propto \rho^{5/3}$ or $P \propto \rho^{4/3}$ for nonrelativistic and relativistic.

So far we've considered completely degenerate material, and we'll go back to that in a minute, but should also mention that a rough boundary between nondegenerate and degenerate is when $E_F > kT$. This allows us to answer the burning question...

For Perspective: am I degenerate? In the old days we'd figure this out by considering my deeds and bad habits, but now we can answer it mathematically! **Ask class:** what do we need to determine? We need to figure out the Fermi energy of my constituents, then compare it to my thermal energy. **Ask class:** if there are plenty of free particles of all kinds, what kind of particle would be degenerate first? Electrons, because they have lower mass and $E_F \propto 1/m$ for nonrelativistic. **Ask class:** is the nonrelativistic limit the correct one? Yes, because 10^6 g cm^{-3} is the rough boundary, and I'm nowhere near that!

In the examples above we've discussed matter that is completely ionized, so that electrons are free to move around as they will. However, in me the electrons are mostly not free. Instead, typically there are ions. So, let's calculate first what the Fermi energy is assuming the dominant species is a molecule of some sort. What is the most common molecule? Water, of course. Water has an atomic weight 18 times that of hydrogen, or about 20 times that of the neutron, roughly speaking. The critical density at which the Fermi energy becomes relativistic goes like M^3, so for water it is about $20^3 \approx 10^4$ times that for neutrons, or about 6×10^{19} g cm^{-3}. Below this density the Fermi energy is nonrelativistic, and therefore goes like $p^2 \sim n^{2/3}$. At my density of ~ 1 g cm^{-3}, the Fermi energy is therefore $\sim 10^{-13}$ times the rest mass energy of water, or $10^{-13} \times 20$ GeV$= 2 \times 10^{-3}$eV. The equivalent temperature for 1 eV is about 10^4 K, so this equates to about 20 K versus about 300 K for the temperature. Sadly, most of my mass is not degenerate! Of course, this is also true for, say, a white dwarf, where the mass is dominated by nondegenerate nucleons but the degenerate electrons provide the pressure.

But there may still be hope for me! Suppose that I have some small fraction of free electrons running around in me. In particular, suppose that there are about 10 electrons per molecule, and that about 1% of molecules have donated 1 electron to the general environment. The density of free electrons is therefore 10^{-3} times the density it would be if all atoms were completely ionized. For the purpose of this calculation, therefore, it's as if I were completely ionized but had a density of about 10^{-3} g cm^{-3}. Using the same approach as before, we know that for electrons the density at which relativistic degeneracy starts is about 10^6 g cm^{-3}, and that below this the Fermi energy scales as $p^2 \sim n^{2/3}$. Therefore, at 10^{-9} of this density the energy is 10^{-6} of the electron rest mass energy, or 0.5 eV. This equates to $\sim$5000 K, meaning that my electrons would be degenerate by a factor of more than 10! Woohoo! Unfortunately, J. Norman Hansen, professor of chemistry and biochemistry, says that in biological systems free electrons essentially don't exist, because as soon as one would be stripped off of a molecule it would go to another one, and hence electrons spend time in one orbital or another. This is also the conclusion we'd reach from the Saha equation: the ionization equation, modulo factors of little interest, would be something like

$$\frac{y^2}{1-y} \propto e^{-3\times10^4/T} \tag{11}$$

for typical electron ionization energies of $\sim$2 eV, meaning that at 300 K the exponential is like e^{-100}, so there is virtually no ionization.

Thus, tragically, I'm not degenerate :). I'm crushed, but let's move on to white dwarfs.

White Dwarfs: Story of discovery of Sirius B (first confirmation of Newton's laws outside the solar system!). Radius measurement from blackbody by W. S. Adams in 1915. Confirmation of gravitational redshift as well, although radius was 5x too large, so the expected effect was 5x *smaller* than the real effect! Classic example of well-intentioned experiments going awry in the expected direction. Great puzzle about how such dense matter could exist, because until that point no one had any idea about degeneracy. Philosophically, this can be regarded as a departure from the principle, established since Newton, that what is in the sky is just an extension of what we have on Earth (e.g., gravity operating the same way). Of course, the physics is the same, but degenerate matter was a novelty. Explained in 1926 by R. H. Fowler.

In 1930, Subramanyan Chandrasekhar worked on the fundamentals of the structure of white dwarfs, coming up with two major results while on a 19-day boat trip from Madras, India, to England to work with Eddington. One was the mass-radius relation $R \sim M^{-1/3}$. The other was completely unexpected: white dwarfs have a maximum mass. He did it by considering, effectively, gamma-law EOS and showing that 4/3 was reached, but a simpler argument is due to Landau.

Ask class along the way: Fermi energy and momentum in relativistic regime, hearken back to uncertainty principle. For relativistic, $E_F \approx pc \sim \hbar n^{1/3} c \sim \hbar c N^{1/3}/R$. That's the energy per fermion that is degenerate. The gravitational energy per baryon of mass m_B at distance R from the center of an object of mass M is **Ask class** about $E_G \sim -GMm_B/R$. If there are roughly the same number of degenerate fermions as baryons, this is also about the gravitational energy per baryon. Adding these together, the total energy per fermion is

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{GNm_B^2}{R} , \tag{12}$$

because the total mass is $M = Nm_B$. Both terms scale as R. Equilibrium is reached at a minimum of the total energy. Suppose first that N is small. Then E is positive, so that E can be decreased by increasing R, and at some point the fermions become nonrelativistic so that $E_F \sim p^2 \sim 1/R^2$, and a stable equilibrium is possible. However, if E is negative then E can be decreased indefinitely by decreasing R, leading to instability and collapse. The boundary point in N is the boundary for stability, and comes at

$$N_{\max} \sim \left(\frac{\hbar c}{Gm_B^2}\right)^{3/2} \sim 2 \times 10^{57} . \tag{13}$$

This gives a mass of about $1.5\,M_\odot$ for the maximum.

Story of Chandra and Eddington. Note that the Landau derivation is a bit fudged, and that the maximum mass predicted in this way is really $1.46(2/\mu_e)^2$, which would be almost $6\,M_\odot$ for neutron stars (GR and strong forces as reasons why not).

Consequences for black holes: from Chandra's work it was known that a white dwarf more massive than about $1.4\,M_\odot$ could not exist, so initially this seemed to doom any star with an initial mass greater than this. However, it turns out that winds and the planetary nebula phase (to be treated later) remove a lot of mass, and up to $\sim 8\,M_\odot$ or so initial mass stars end up as white dwarfs. Higher than that and something else happens. Chadwick discovered neutron in 1932. Almost immediately, Landau realized the possibility of "neutron stars" which might save the day. In 1934, Baade and Zwicky suggested that supernovae might be the transition from normal stars to neutron stars. It seems, however, that above an initial mass of $\sim 30\,M_\odot$ (quite uncertain) fallback will produce a black hole (of course, numerical simulations can't even get SN to explode, so...). This has been aesthetically displeasing to many people, but is accepted by almost all now.