Equations of state, introduction

Did well on homework. Some common mistakes: in problem 1, many got the full period (2π) of the free-fall oscillation. But, to compare to the other time scales needed half of this period. Argument about what would happen for a centrally condensed star: many said $c_s \propto \rho^{-1/2}$, but remember that it is also proportional to $P^{1/2}$, so this argument is not clean. Free-fall time is clearly less for a centrally condensed star, implying (correctly) that a more condensed star has a shorter period, all else being equal.

In problem 3, several people used $M_r = (4\pi/3)r^3\rho$. Remember, though, that the actual relation is $dM_r = 4\pi r^2 \rho dr$. Can't just integrate this. For example: the outer layers of the Sun (e.g., photosphere) have extremely low ρ, so certainly $M \neq (4\pi/3)R^3\rho_{\rm photosphere}$!

Problem 4b, many people said it took them a long time to get the reference. How did they search? Are there hints I can give?

The equation of state, in general, is the relation between pressure, density, and temperature. Included in this is the idea of the ionization level. **Ask class:** what are states of matter in their everyday experience? Gas, liquid, solid. **Ask class:** what is it that determines whether a material is a gas, liquid, or solid? Composition, density, and temperature matter. **Ask class:** from a more basic standpoint, what quantities might one compare to determine if something is, e.g., a gas or a liquid? Temperature versus interaction energies might be a good one. For solid, need crystallization, so again compare temperature with interaction energy. Will get criterion later.

Note: in the preceding discussion we've implicitly assumed that matter will reach some kind of equilibrium state. That means that there must be enough time to do this. **Ask class:** is the interior of the Sun in equilibrium? If not, in what ways is it in equilibrium and in what ways is it not in equilibrium? It certainly is not in nuclear equilibrium, because it is undergoing fusion! Is it in thermal equilibrium? How would we know? Over large scales it definitely is *not* in thermal equilibrium (15 million K at center, 5,800 K at photosphere, in contact with a 3 K thermal bath). Can we simplify locally, though? **Ask class:** suppose we want to argue that local thermodynamic equilibrium (LTE) holds to a good approximation. What do we have to show? We need to show that over a distance comparable to a mean free path of a photon, relevant quantities change very little. **Ask class:** what are those relevant quantities? Temperature, pressure, density.

Calculate this: pressure scale height is

$$\lambda_P = -\left(\frac{d\ln P}{dr}\right)^{-1} = \frac{P}{g\rho} = \frac{R^2}{2r}\left[1 - \left(\frac{r}{R}\right)^2\right] , \tag{1}$$

where the last equality holds for a constant-density star. This is infinite at the center and of order R for most of the star. More generally, from the equation of hydrostatic equilibrium we know that $dP/dr = -\rho(GM_r/r^2)$. We can manipulate this with dimensional estimates to show that again $\lambda_P \sim R$ for most of the star. The temperature gradient we already went through: if we take 10^7 K over 10^{11} cm as an approximate value, that's 10^{-4} K cm^{-1}. By comparison (**ask class**), the distance typically traveled by a photon is $(n\sigma)^{-1} = (\kappa\rho)^{-1}$, or about 1 cm.

Thus, although there is energy flow, it occurs on large length scales and locally, thermodynamic equilibrium is usually an excellent approximation. **Ask class:** what are some circumstances in which this might not be a good approximation? (supernova, nova, possibly large-scale convection). Also not a great approximation anywhere outside the stellar photosphere, where the material is optically thin (e.g., the corona). Is LTE a reasonable approximation in the classroom? Cold air from outside, hot air from professors. Again, need $(n\sigma)^{-1}$, which is about $(10^{20}10^{-16})^{-1} = 10^{-4}$ cm, so probably yes.

Chemical potential:

$$\mu_i = \left(\frac{\partial E}{\partial N_i}\right)_{S,V} . \tag{2}$$

Here N_i is the number density per gram of the ith species, so that $N_i = n_i/\rho$. Thermodynamic and chemical equilibrium require that if there are reactions that might change the N_i, then $\sum_i \mu_i dN_i = 0$. Photon number is not conserved, so $\mu_\gamma = 0$. Note that here we're interested in reactions that take place fairly rapidly, so things like nuclear reactions (which usually take years to billions of years) aren't included. Technically, these reactions mean that the system is *not* in equilibrium, but this is another example of how we simplify by dropping small terms. However, we should remember that it is possible to get to a lower-energy state via these reactions, and that at some point if the reactions are fast compared to chemical reactions then they must be included.

The dN_i are related to each other by particular reactions, so that means we can also write $\sum_i \mu_i\nu_i = 0$, where ν_i are the stoichiometric coefficients. From this, can see in another way that $\mu_\gamma = 0$: if a reaction of the form $A \rightarrow B + \gamma$ is allowed, so is $A \rightarrow B + \gamma + \gamma$. If μ_γ were nonzero, equilibrium would not be possible.

A lot of the interesting thermodynamic quantities can be calculated from the distribution function in 6-dimensional phase space (three coordinate, three momentum). This is also called the occupation number. For a given species, the distribution function for a particular momentum is

$$n(p) = \frac{1}{h^3} \sum_{\rm states} \frac{1}{e^{E_{\rm tot}({\rm state})/{\rm kT}} \pm 1} . \tag{3}$$

Here $E_{\rm tot} = -\mu + E({\rm state}) + {\rm E(p)}$, where E(state) is the energy of the state of that species relative to some level (often the ground state energy) and $E(p)$ is the kinetic energy for momentum p. Note that the energy level relative to which one determines $E_{\rm state}$ is a free parameter, but $-\mu + E$state isn't, which can lead to varying definitions for μ (beware!). For degenerate energy states the distribution function is sometimes written with a g_j in the numerator, which is the number of states having the same energy E_j. Same deal. Of course (**ask class**) the ± 1 refers to Fermi or Bose statistics. The point is that Fermi wavefunctions (for half-integral spin particles) are antisymmetric, so they avoid each other and the "occupation number" (the $1/[\exp(blah) - 1]$ bit in the distribution function) is always less than 1. Bose wavefunctions (for integral spin particles) are symmetric, and can have occupation numbers from 0 to infinity. **Ask class:** what are some examples of fermions? Electrons, protons, neutrons, for example (also neutrinos, quarks). **Ask class:** what are some examples of bosons? Photons are the best known among the elementary particles. Also, vector bosons (Z^0, $W^\pm$, carriers of the electroweak force), gravitons (spin 2, if they exist), and I believe gluons. Interesting that force carriers are bosons, and the "point particles" are fermions.

The distribution function is in (cm-momentum)$^{-3}$ units. Usually assume that momentum space is spherically symmetric, so that the physical number density is

$$n = \int_p n(p) 4\pi p^2 \, dp \; . \tag{4}$$

In general, the kinetic energy is **Ask class:** $E(p) = (p^2c^2 + m^2c^4)^{1/2} - mc^2$. The isotropic pressure is

$$P = \frac{1}{3}\int_p n(p) p v 4\pi p^2 \, dp \tag{5}$$

where the velocity is $\partial E/\partial p$, and the internal energy is

$$E = \int_p n(p) E(p) 4\pi p^2 \, dp \; . \tag{6}$$

Tell class: we are now going to concentrate for a while on "perfect" noninteracting particles, for simplicity. Later, we will think about the interactions of particles with other particles, which is what produces the liquid and solid states. Still later, we will talk about really high densities and some of the things that can arise in white dwarfs and neutron stars.

Specific application: blackbody radiation. For photons, $\mu = 0$, $g = 2$ (because there are two polarizations), $E_j = 0$ (because there are no excited states), $E(p) = pc$, and minus in the denominator because photons are bosons. Put it together and what have you got? (bippity boppity boo, but never mind that). The photon number density is then

$$n_\gamma = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 \, dp}{\exp(pc/kT) - 1} \approx 20\, T^3 \ {\rm cm}^{-3} \; . \tag{7}$$

The radiation pressure is $aT^4/3$ and the energy density is aT^4, where $a = 8\pi^5 k^4/15c^3h^3 = 7.6 \times 10^{-15}$ erg cm^{-3} K^{-4}. **Ask class**: we talked about gamma-law equations of state earlier. What does this mean for gamma? ($P = (\gamma - 1)E$, so $\gamma = 4/3$). What consequence does that have for a star whose pressure and energy density are dominated by radiation? Close to instability. This is one of the reasons that very high-mass stars $M > 100\,M_\odot$ are not very stable. For somewhat separate reasons having to do with cooling, accretion disks that are dominated by radiation pressure are also unstable.

From this, we can also get the spectral distribution for a blackbody.

$$u_\nu \, d\nu = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \text{ erg cm}^{-3} \text{ Hz}^{-1} \text{ Hz} . \tag{8}$$

Monatomic Gas: The bit about $\mu \ll 0$ for an ideal gas can be derived by computing the number density and then getting μ from it; can assume it now and find that it is correct.

Given that it is, the ± 1 in the denominator is superfluous. Assume a nonrelativistic gas for starters, so **Ask class**: $E(p) = p^2/2m$ and $v = p/m$. Assume only one energy level, $E = E_0$; we can reference the energy to this single energy level and define $E_0 = 0$ (this simply redefines μ relative to that energy level, and different definitions exist in the literature). For an ideal gas $\mu \ll -kT$. The number density is then

$$n = \frac{4\pi}{h^3} g \int_0^\infty p^2 e^{\mu/kT} e^{-p^2/2mkT} \, dp . \tag{9}$$

This can be integrated to find the relation between μ and n, the total number density. This relation is

$$e^{\mu/kT} = \frac{nh^3}{g(2\pi mkT)^{3/2}} . \tag{10}$$

If you have a state with an energy E_j relative to the reference energy, then with this simplification the equation above would have to be multiplied by $e^{E_j/kT}$ on the rhs.

Similar integrations show that $P = nkT$ (big surprise!) and $E = \frac{3}{2}nkT$. What is gamma-law for this equation of state? ($P = (\gamma - 1)E$, so $\gamma = 5/3$).

Saha equation: This is *thermal* ionization. Consider a hydrogen atom with a single energy level (ground state), and consider the ionization reaction

$$H^+ + e^- \longleftrightarrow H^0 + \chi_H , \tag{11}$$

where χ_H=13.6 eV is the ground state binding energy. In equilibrium, what can be said about the abundances? In this case, let the reference energy ($E = 0$) be for the free electron and proton. Statistical weights: 2 for the hydrogen atom, two combined for the proton and electron (spins in same direction, spins in opposite directions). The number densities are then

$$n_e = \frac{2[2\pi m_e kT]^{3/2}}{h^3} e^{\mu^-/kT} , \tag{12}$$

$$n^+ = \frac{[2\pi m_p kT]^{3/2}}{h^3} e^{\mu^+/kT} , \tag{13}$$

and

$$n^- = \frac{2[2\pi(m_p+m_e)kT]^{3/2}}{h^3} e^{\mu^0/kT} e^{\chi_H/kT} \ . \tag{14}$$

Ask class: if we were being *really* accurate, would the actual mass in the third equation be $m_p + m_e$? (No, because we'd have to subtract the binding energy. But this is only 13.6/511,000 or a 1/40,000 correction, and can be dumped. In the same spirit, the electron mass is only a 1/1,800 correction and can be ignored here.)

Forming the product $n^+ n_e / n^0$, we find

$$\frac{n^+ n_e}{n^0} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{(\mu^- + \mu^+ - \mu_0)}/kT e^{-\chi_H/kT} \ . \tag{15}$$

Ask class: what is $(\mu^- + \mu^+ - \mu_0)$ in equilibrium, which we've been assuming? It's zero. The numerical factor is about $2.4 \times 10^{15} T^{3/2}$ cm^{-3}. **Ask class**: is this equation sufficient to determine the abundances of electrons, protons, and neutral hydrogen? No, because we have only one equation for three unknowns. **Ask class**: what are reasonable auxilliary conditions? Charge balance implies $n^+ = n^-$, and conservation of nucleons means that $n^0 + n^+ = n$, the total number density. With these extra conditions, the degree of ionization $y \equiv n^+/n = n_e/n$ is, numerically,

$$\frac{y^2}{1-y} = \frac{4 \times 10^{-9}}{\rho} T^{3/2} e^{-1.6 \times 10^5/T} \ , \tag{16}$$

where as usual quantities such as ρ and T are referenced to the appropriate cgs unit.

Ask class: what does this mean about the degree of ionization for a fixed temperature but varying density? (degree of ionization goes up when density goes down). Why is that? This is thermal ionization, so you can imagine some radiation field pervading the region. The ionization rate per volume, therefore, just goes like the number density. However, the recombination rate involves two particles, and hence must go as the product of their densities. Therefore, the recombination rate increases more rapidly when the density goes up, and the resulting ionization fraction goes down.

Ask class to compute ionization fraction in the center of the Sun. Suppose **Ask class** $T = 1.5 \times 10^7$ K and $\rho = 100$ g cm^{-3}, for round numbers. Then $y^2/(1-y) = 2.3$. This has a solution of $y = 0.75$, so the center of the Sun is 25% neutral (!). **See if this poses a problem for anyone in class!**. This is nonsense, of course. Resolution is that at such densities, the separation between atoms is much less than the size of an individual free atom, so the electrons get squeezed up to continuum states and *pressure* ionization dominates. Go back to the original Saha equation and indicate that what really should have been included was *all* energy states of the atom, and that this sum would diverge because there are in principle an infinite number of finite terms that would have to be summed in this way. In reality, the size of the orbital increases like n^2, so in most circumstances only the first term needs to be kept, as we did.

Idea of degeneracy. Note, however, that uncertainty principle means that even for bosons the energy goes up with increasing density, and so provides support. At some point I do want to go into topics appropriate for high density, i.e., white dwarfs and neutron stars.