Basics of Thermonuclear fusion

Energetics. Define binding energy. Binding energy curve (H=0, He=7.1, C=7.7, Fe=8.8 MeV, all per nucleon. Drops to 7.5 near Uranium; see pg. 215). Approximate density (about 1.5×10^{14} g cm^{-3}). Coulomb barrier: distance is about 1 fm, so $e^2/10^{-13}$ cm=1.4 MeV. Maxwell-Boltzmann curve, exponential part is $e^{-E/kT}$, which is about e^{-1000} at the Sun's temperature! Need tunneling.

Ask class: suppose Dr. Evil prevented tunneling, so that the only way to fuse would be classically (i.e., you'd have to have at least some nucleons with energies above 1.4 MeV). What would happen to the Sun? It would shrink by a lot, until the central temperature got high enough that some fusion could happen.

Ask class: let's suppose that for some reason you have an extremely dense fluid of hydrogen, which is very cold. Can fusion take place then? Yes, if the nuclei are close enough to each other then their wavefunctions overlap enough that tunneling is possible even without temperature. Another way to look at this is that for very dense matter there is an extra contribution to the energy, called the Fermi energy. As we will see later (and as they may have seen before), the Fermi energy is (among other things) the energy below which there is almost complete occupation by fermions. This can be related to a Fermi momentum using the uncertainty principle:

$$\Delta p \Delta x \sim \hbar \; . \tag{1}$$

This means, more or less, that the momentum cannot be less than about $\hbar/\Delta x$. Thus, if some species of particle has a number density n, so that $\Delta x \sim n^{-1/3}$, then the momentum isn't less than $p_F \sim \hbar n^{1/3}$. If this is nonrelativistic, then $E_F \approx p_F^2/2m$, and if relativistic then $E_F \approx p_F c$. This is real energy, in the sense that the energetic "cost" of having a particle includes the Fermi energy. This is not important for main sequence stars, but is for stellar remnants.

We can now qualitatively understand two things: the general structure of the binding energy curve, plus some things about neutron stars.

Try to lead class through this: Like many things, the structure of nuclei is determined by what gives a minimization of energy. **Ask class:** what are the contributions to that energy? A negative (binding) contribution from the strong force; a positive (repulsive) contribution from the electrostatic force; and a positive (repulsive) contribution from the Fermi energy. An essential point is that the strong force has a very short range (of order a Fermi, 10^{-13} cm), whereas the electrostatic force has an infinite range (by which I mean that it doesn't die off exponentially). Starting off with small nuclei, adding nucleons increases the binding energy, making the nucleus more stable. When the nucleus gets large enough, however, the electrostatic repulsion works through the whole volume, whereas the strong force is only local, so electrostatic repulsion starts to win and the binding energy per nucleon decreases (it does not become positive, but it decreases). This can lead to instability and radiation such as alpha-particle, or fission. The peak in the binding energy curve at zero pressure is ^{56}Fe. **Ask class:** given that, why isn't everything pure iron? Potential barrier is too large to penetrate. This is an example of a nonequilibrium situation. This happens in other cases, as well; for example, the temperature of the universe is about 3 K, yet the intergalactic medium is highly ionized. **Ask class:** how can this be? Same thing. When the reaction rates are sufficiently slow, systems can be dramatically out of equilibrium. Some people have suggested that the true ground state of matter is as strange matter, but again the potential barrier is too high in most cases to reach this state.

Could also mention nuclear magic numbers (don't ask them to derive); recent synthesis of ^{48}Ni, which is doubly magic but ridiculously proton-rich.

In neutron stars, as density gets higher the Fermi energy of electrons increases more rapidly than of p or n (because of m in denominator). Becomes relativistic at about 10^6 g cm^{-3}. It becomes energetically less favorable to have both p and e$^-$ (two particles) versus n, so equilibrium nuclei at high densities are extremely neutron-rich: e.g., 40 protons, 80 neutrons. At a density of about 4×10^{11} g cm^{-3}, free neutrons are stable and you have "neutron drip" in which free neutrons coexist with nuclei. In core, have few$\times 10^{14}$ g cm^{-3}, and could have $\sim$90% neutrons plus charge-balance of p and e, or (for same Fermi energy reasons) could have other exotic particles or states of matter.

Later, perhaps during the lecture on heavy element burning, I need to point out (Clayton pg 519) that when you get to really heavy elements, the Coulomb barrier is large enough that you have lots of photodisintegration and hence rearrangement of the elements. There is also substantial neutrino emission above 3e9 K, so the net rate of energy generation is small.

Thermonuclear fusion, continued

With this lecture, extremely important to have 5 minutes at the end to recap the main points!

Nomenclature: A reaction of the form

$$a + X \to Y + b \tag{2}$$

is often represented $X(a, b)Y$. An intermediate state of an excited nucleus is represented with an asterisk: Z^*, or a second type of excited nucleus could be Z^{**}. There might be such an intermediate state, as in

$$p + ^{11}B \rightarrow ^{12}C^* \,. \tag{3}$$

Here the number ^{11} in front of B means the atomic weight. The excited state, or compound state, often has many ways in which it can break up. For example, $^{12}C^*$ can break up into $^{12}C^{**} + \gamma$ (a lower excited state plus a photon), $^{11}B + p$ (the original combination of boron and a proton), $^{11}C + n$, and so on. In all cases, when considered in isolation the reactions must conserve **Ask class:** (1) energy, (2) linear momentum, (3) charge, (4) baryon number, and (5) lepton number (separately for electron, muon, tau). For these last two, antiparticles count as -1. Angular momentum must also be conserved, but that's more detailed. For example, for neutron decay in free space, $n \rightarrow p + e^- + \nu$, **Ask class:** what kind of a neutrino must it be? An electron antineutrino, to conserve lepton number. The reaction $\gamma \rightarrow \gamma + \gamma$ in free space satisfies these conservation laws. **Ask class:** can it happen? No, it is kinematically forbidden because the two photons must be precisely parallel to conserve momentum, and therefore there is zero solid angle in which this can take place. **Ask class:** when we talk about energy conservation, what must be included? The full energy, including rest mass energy, Fermi energy, and whatnot. **Mention: conservation of energy related to invariance with time; conservation of linear momentum related to homogeneity; conservation of angular momentum related to isotropy. GR has no global energy conservation; time now not same as near big bang!**

For each mode i, can define a mean lifetime for decay τ_i. **Ask class:** for such a lifetime, how do we estimate the width in energy from the uncertainty principle? Simply $\Gamma_i = \hbar/\tau_i$ in energy. **Ask class:** given several channels, which will likely occur first, a long-lived channel or a short-lived channel? The short-lived channel will likely happen first, so it's more probable. **Ask class:** given a number of possible decay channels with lifetimes τ_j, what is the probability of decay through a particular channel i? Just $(1/\tau_i)/\sum_j(1/\tau_j) = \tau/\tau_i$, where $\tau = 1/\sum_j(1/\tau_j)$ is the total mean lifetime. This can therefore also be written $P_i = \Gamma_i/\Gamma$. Note that this assumes a Poisson distribution: for a single channel, the probability that a decay will occur after a time t is $1 - \exp(-t/\tau_i)$. Note also that this type of *harmonic averaging* (weighting by the reciprocal) will also come up in opacities.

The Γ_i can be broken into two factors: a probability that the state is composed of the desired constituents, which is represented by the "reduced width" γ_i^2, and a Coulomb penetration factor, which allows those constituents to get to infinity. Usually written $\Gamma_i = 2P_\ell\gamma_i^2$.

Cross sections.—In everyday life, the cross section of something is often simply its geometrical area. If you throw projectiles at me, the ones that hit will hit in my area in profile, maybe 0.6 m^2 face-on, and half that edge-on. As long as the projectile is small compared to me, its nature doesn't matter much; a water balloon or a rotten tomato will have the same effective cross section of interaction. If we wanted to, we could define my cross section in another way. Suppose there were to be a steady pelting by rotten tomatoes, but that everyone had such bad aim that any area had the same probability of being hit as any other area. We can then define the tomato flux, $F_{\rm tom}$, as the number per area per second. My cross section is then $R_{\rm tom}/F_{\rm tom}$, where $R_{\rm tom}$ is the rate at which they hit me in number per second.

The cross section is not always independent of the nature of the projectiles. **Ask class:** if the "projectiles" are photons, can they think of an everyday example of this? A pane of glass is nearly transparent to visible light, but is opaque to infrared, hence the greenhouse effect. The effective cross section is defined as rate divided by flux, again, making the effective cross section much less for visible light than for infrared. Similarly, for some reaction $X+\alpha\rightarrow Y+\beta$, we define a cross section $\sigma_{\alpha\beta}(v)$ as the number of such reactions per second per target divided by the incident flux of the projectiles. Then the reaction rate per volume is $\sigma_{\alpha\beta}(v)vn_\alpha n_X$ cm^{-3} s^{-1}. If the target and projectile are the same, have to divide by two to avoid double-counting. **Ask class:** can they think about how to get the average rate if there is a distribution of particle velocities? Have to integrate $\sigma_{\alpha\beta}v$ over the particles, which are distributed in a Maxwell-Boltzmann distribution. In center of mass frame, the relative velocities are also distributed that way, so can get the average as in eqn. (6.26) of HK:

$$\langle\sigma v\rangle_{\alpha\beta}=\left(\frac{8}{\pi m}\right)^{1/2}(kT)^{-3/2}\int_0^\infty\sigma_{\alpha\beta}(E)e^{-E/kT}E\,dE\ . \tag{4}$$

The cross section can be written in the form (6.28):

$$\sigma_{\alpha\beta}(E)=\pi\bar{\lambda}^2 g\frac{\Gamma_\alpha\Gamma_\beta}{\Gamma^2}f(E)\ . \tag{5}$$

Factors break down as follows: $\bar{\lambda}$ (which should really have the bar across the lambda, not on top of it) is the reduced DeBroglie wavelength, so that $\pi\bar{\lambda}^2=\pi\hbar^2/2Em$. This, roughly, is what the cross section would be for a perfect absorber. g is a statistical factor. $\Gamma_\alpha\Gamma_\beta/\Gamma^2$ is the probability that the initial particles will get together multiplied by the probability that the end products will be as specified. The final factor, $f(E)$, is the "shape factor" and depends on the process: can be "resonant" or "non-resonant". Resonant is peaked at some energy (see 6.30 in HK), non-resonant either is intrinsically non-resonant or is well below the peak.

Non-resonant: **Ask class:** qualitatively, how do they expect the probability of barrier penetration to depend on the electric charges of the reactants? For all else equal, must decrease with increasing charge because the Coulomb repulsion is greater. **Ask class:** for all else equal, how should barrier penetration depend on the energy of the particle? Should be more probable for higher energy. From this, the barrier penetration factor is $P_l(E)\propto e^{-2\pi\eta}$, where $\eta=Z_\alpha Z_X e^2/\hbar v$, or $0.1574Z_\alpha Z_X(\mu/E)^{1/2}$ (μ is the reduced mass, $\mu=m_\alpha m_X/(m_\alpha+m_X)$). Putting this together, the cross section for non-resonant interactions is like

$$\sigma_{\alpha\beta} = \frac{S(E)}{E} e^{-2\pi\eta} . \tag{6}$$

Net result is that $\langle\sigma v\rangle$ is proportional to $\int_0^\infty S(E)\exp\left[-\left(\frac{E}{kT}+\frac{b}{E^{1/2}}\right)\right] dE$, where $b \equiv 0.99 Z_\alpha Z_X \mu^{1/2}$ (MeV)$^{1/2}$. Clearly, there is a tradeoff between barrier penetration (increasing with increasing energy) and number of available particles (decreasing with increasing energy). The maximum of the integrand is called the "Gamow peak": $E_0 = 1.22(Z_\alpha^2 Z_X^2 \mu T_6^2)^{1/3}$ keV. Full width at $1/e$ max is $\Delta \approx 2.3(E_0 kT)^{1/2}$. For example, at 2.2×10^7 K and using the p+^{12}C reaction, $E_0 = 30.8$ keV and $\Delta = 17.6$ keV. This reaction is extremely sensitive to temperature: the peak at 2.2×10^7 K is 5x the peak at 2.0×10^7 K.

The shape factor $S(E)$ has to be determined experimentally. Over the range of interest it is close to constant. Integration of this expression isn't possible in closed form. One trick is to integrate a Gaussian that is of the same height and curvature at maximum, and this works reasonably well because the integrand is strongly peaked.

Point out that the reactions in the Sun are extremely rare at the given temperature. How do we know? Figure out amount of hydrogen that is being converted to helium per time, relate this to the number of possible interactions. Not much at all. Extrapolation to much lower energies, but we still have some confidence because the theoretical curve fits well. Still want to be alert, though. **Ask class:** given that the reactions are rare, what does this mean about temperature sensitivity? Since the reactions are really exponential, it implies that there is *tremendous* sensitivity to temperature. A small change in the temperature can affect the energy generation rate a great deal. **Ask class:** so what can we say about how the central temperature of stars burning the same fuel in the same way depends on mass? It is quite insensitive.

Bring them back to the point that these reactions are inherently exponential in the temperature, but that it is analytically convenient to express things in power law form. Therefore, in the form $\epsilon = \epsilon_0 \rho^\lambda T^\nu$ that we used before, the logarithmic derivatives are $\lambda = 1$ and $\nu \approx 14(Z_\alpha Z_X \mu)^{1/3}/T_6^{1/3} - 2/3$. For example, $^{12}C(p,\gamma)^{13}N$ at $T = 2\times10^6$ K has $\nu = 16$.

Resonant reactions: Near the peak energy of a resonant reaction, the cross section is dominated by the $[(E-E_r)^2 + (\Gamma/2)^2]^{-1}$ Lorentzian form. Integration of this is relatively straightforward because of the peaked nature of the cross section. The net temperature exponent is $\nu = 11.61 E_r/T_9 - 3/2$.

Neutron capture cross sections: **Ask class what the major difference is (no Coulomb barrier!)**. Not important for energy production, but it is important for figuring out how heavy elements got to be the abundances they have. s process: capture is slow compared to beta decay. r process: capture is fast, so more neutron-rich elements can be produced (only in supernovae).

Electron screening: a cloud of electrons will gather around the positively charged nuclei, reducing the Coulomb barrier and increasing the reaction rate. For small corrections of this type, this ends up increasing the rate by a factor $\exp(U_0/kT)$, where U_0 is a screening energy. For protons onto ^{12}C at 2×10^7 K, the factor is $\exp(U_0/kT) \approx 1.25$.

Pycnonuclear reactions: at high densities, nuclear wavefunctions overlap enough that even at zero temperature fusion can occur. This requires densities only found in a white dwarf or neutron star, of order 10^{6-9} g cm^{-3}. **Ask class:** are these sensitive to temperature? No, very temperature insensitive until the thermal energy is comparable to the Fermi energy. Can also think about this as Fermi energy substituting for temperature.

Stability versus instability?