Equations of Stellar Structure

We're getting closer to being able to construct simple models of stars. To do this we need to write a couple more equations relating various quantities; an energy transfer equation, which relates the gradient of the energy density to the luminosity, and an equation of state to relate the pressure to the density and temperature. Once we have that, we have a complete set of equations (in principle) and will apply them in a simple model. In the next several classes we will then focus on particular aspects of these relations: energy generation, equations of state, and opacities.

For energy transfer, we'd like to think about the *net flux* of energy a given location. Since we're restricting things to spherical symmetry, **Ask class:** what is the only direction in which such flux can be nonzero? The radial direction. In the interior of stars (which is what we care about), the mean free path is small enough that we can think about diffusion (i.e., particles scatter or interact an enormous number of times before they leave the star). Thus, the flux depends in some way on the energy density. **Ask class:** if the energy density is constant throughout the star, will there be net flux? No, because as much goes down as up. So, **Ask class:** how must the flux be related to the energy density? Through the gradient of the energy density, which is dE/dr since spherical symmetry applies. Now, let's say that the energy is transported by some particle with a mean free path ℓ. **Ask class:** for all else equal, will the net flux be larger if ℓ is larger or smaller? Larger, because a larger change in energy density is sampled. Conversely, if the opacity κ is large (meaning that the mean free path is small), the net flux must be small. Therefore, $F \sim \kappa^{-1}dE/dr$. **Ask class:** if r and F are both positive in the outward direction, is the sign positive or negative in the equation for F? Must be negative.

In main sequence stars we usually think about radiation energy density being the most important thing, largely because in those circumstances photons travel farther than electrons or other particles and are therefore most effective at carrying energy (this is not, however, true in degenerate material). The energy density of radiation is aT^4, and when various other factors are put in (see chapter 4), the flux is $F_r = -(c/3\kappa\rho)d(aT^4)/dr$, so that the luminosity is

$$L_r = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr}\ . \tag{1}$$

This may also be written in the Lagrangian form (i.e., with dM_r and the fundamental differential instead of dr) using the relation $dM_r = 4\pi r^2\rho dr$.

The opacity has dimensions of $\mathrm{cm}^2\mathrm{g}^{-1}$, and is therefore the total cross section of a gram of material. We'll talk about the opacity in much greater detail later, but for now note that $1/(\kappa\rho)$ has dimensions of centimeters, and is in fact. the mean free path. The factor of c on the top is because radiation travels at the speed of light; clearly, the faster your energy-carrying particles move, the more flux is transported, for all else equal. The full calculation of opacities can be awful. Luckily, for many applications one can write it as

$$\kappa = \kappa_0 \rho^n T^{-s} \ , \tag{2}$$

where κ_0, n, and s are constants. For example, for Thomson scattering $n = s = 0$, whereas for a Kramer's law of opacity $n = 1$ and $s = 3.5$, which is useful for various atomic opacities. The final equation we'd like is the equation of state, relating pressure to density and temperature. In the same spirit as the opacity, we can write

$$P = P_0 \rho^{\chi_\rho} T^{\chi_T} \ , \tag{3}$$

with P_0, χ_ρ, and χ_T constant. For example, for an ideal gas $P = nkT$ so $\chi_\rho = \chi_T = 1$. Similarly, the energy generation rate ϵ (such that $dL/dM = \epsilon$) is often written as $\epsilon = \epsilon_0 \rho^\lambda T^\nu$. Note that all these power-law forms are approximations (albeit surprisingly good ones for many applications), and that they also really depend on things such as the composition.

We can put these all together in our four fundamental equations of stellar structure:

$$\begin{array}{ccc} & \text{Eulerian} & \text{Lagrangian} \\ \text{Mass equation} & M(r<r_0) = \int_0^{r_0} \rho 4\pi r^2\, dr & M(r<r_0) = \int_0^{r_0} \rho 4\pi r^2\, dr \\ \text{Hydrostatic equation} & dP/dr = -\rho GM/r^2 & dP/dM = -GM/4\pi r^4 \\ \text{Energy balance} & dL/dr = 4\pi r^2 \rho\epsilon & dL/dM = \epsilon \\ \text{Energy transfer} & L = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr} & L = -\frac{(4\pi r^2)^2 c}{3\kappa}\frac{d(aT^4)}{dM} \end{array} \tag{4}$$

Dimensional Analysis and Homology Relations

The idea of homology relations is to suppose that there is a single "standard star" and that all other stars are scaled versions of this. Among other things this means that we assume spherical models with the same uniform composition and microphysics. **Note:** this means that there is no preferred scale; all stars are intrinsically alike. This implies that various functions are power laws, since these have no scale either (true *only* of pure power laws, in the sense that, e.g., if $f(2x)/f(x)$ is independent of x, f must be a pure power law). Example: Newtonian orbit of two bodies. With no time or length scale, **Ask class:** what is the only thing we know about these? Mass ratio. With GR precession, get $M_{\rm tot}/a$, with gravitational radiation get $a^4/(\mu M_{\rm tot}^2)$, so have everything.

Lesson: for scale-free quantities, power laws rule. Whenever there is a preferred scale, the power laws are broken. We'll keep this in mind because for homology relations we will assume power laws, but they are valid only as long as the scale free property works (e.g., the microphysics is the same for all the stars in our sample).

Thus, assume that r and M_r in a star of radius R and mass M are related to each other, with respect to a reference star of radius R_0 and mass M_0: if

$$r = \frac{R}{R_0} r_0 \tag{5}$$

then

$$M_r = \frac{M}{M_0} M_{r0,0} \ . \tag{6}$$

That is, the mass interior to r in the general star is equal to M/M_0 times the mass interior to $r0$ in the reference star.

Now derive the equations relating the derivatives of log quantities to each other, using this Lagrangian formalism. We'll divide out the common factor $d \ln M$. Start with mass equation. **Ask class:** do we need to retain constant factors like G, c, R_0, and M_0, given that we are going to take the log and then the derivative? No, because these become zero. (**Note:** do the mass equation, then ask class to select one more, but don't do all.)

1. Mass equation. $\rho R^3 \propto M$. Take log, then derivative, and get $d \ln \rho + 3d \ln R = d \ln M$. Dividing out, we get $d \ln \rho / d \ln M + 3d \ln R / d \ln M = 1$. That means that if we assume power-law relations in the form $R \propto M^{\alpha_R}$ and $\rho \propto M^{\alpha_\rho}$, we get $\alpha_\rho + 3\alpha_R = 1$.

2. Equation of hydrostatic equilibrium. Integrate, drop constant factors, and find $P \propto M^2/R^4$, where P is the central pressure. Assume the pressure can be written in the form $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$ (this is the equation of state) and that χ_ρ and χ_T are the same for all the stars in the set. Taking the logarithmic derivative, we get

$$4d \ln R + \chi_\rho d \ln \rho + \chi_T d \ln T = 2d \ln M \ . \tag{7}$$

Having done this it should not be necessary to do the rest. It should be possible to simply say we're also taking $T \propto M^{\alpha_T}$ and $L \propto M^{\alpha_L}$. Just in case, however:

3. Energy equation. Integrate, get $L \propto \epsilon M$, so that $d \ln L - \lambda d \ln \rho - \nu d \ln T = d \ln M$.

4. Transfer equation. Integrate, drop constants, get $LM \propto R^4 T^4 \kappa^{-1}$, so that the logarithmic derivative equation is $d \ln M = -d \ln L + 4d \ln R + (4 + s)d \ln T - nd \ln \rho$.

Mention that final matrix is in H+K. Don't solve or put in the solution, have students look it up. Mention (but do not do) convection. For moderately massive stars, where electron scattering dominates ($n = s = 0$ for the opacity), the equation of state is close to that of an ideal gas ($\chi_\rho = \chi_T = 1$), and energy generation is supplied by the CNO cycle ($\lambda = 1$, $\nu \approx 15$), HK derive that the expected radius is $(R/R_\odot) = (M/M_\odot)^{0.78}$ and the expected luminosity is $(L/L_\odot) = (M/M_\odot)^3$. The empirical exponents in this realm are 0.75 and 3.5, respectively, so the homology analysis has done well. It would *not* do well for much lower-mass stars, where the microphysics is different.

Ask class: given that during the main sequence stars convert a nearly fixed fraction (about 15%) of their hydrogen to helium, how would we use the luminosity relation to estimate the MS lifetime of stars, relative to that of the Sun? Total mass available to burn is M, so the total energy available is also proportional to M and the time is thus proportional to M/L, so $T = T_\odot[(M/M_\odot)/(L/L_\odot)]$, or $T \approx 10^{10}(M/M_\odot)^{-2.5}$ yr.

Observables of Stars, Conventions

What can be directly observed for a distant star? Total brightness, spectrum. For close stars, distance can be measured with parallax or more indirect methods. Introduce idea of spectral bands: B, V, R, I, and so on. Complications of different filters! (but don't confuse them too much with this; bring along Harwit). Apparent magnitude: bands relative to Vega, which is defined as magnitude 0 in all bands. Logarithmic system: 5 magnitudes equals a factor of 100 in flux in a given band. If a star is *dimmer*, its magnitude is *greater*; therefore, something with magnitude +5 in the V band is 100 times dimmer than Vega. Historical antecedent is probably that people would talk about "a star of the first magnitude" being brighter than one of the second magnitude, and probably the proportions between magnitudes are about the level distinguishable by eye. Dark site, good vision, magnitude 6 is the typical naked eye limit. Brightest star in night sky is Sirius, at magnitude -1.5.

Absolute magnitude: magnitude that star would appear to have if it were 10 pc from us (make sure they know what a parsec is!). The Sun has an absolute V magnitude of 4.8. From this, ask the class to calculate its apparent V magnitude (obviously they need the Sun's distance from us and the inverse square law). Also: given that Sirius has an apparent magnitude of -1.5 and an absolute magnitude of 1.4, how far away is it? (answer: 8.6 lyr, or 2.6 pc). Note that this is one way of determining the distance of an object: knowing its true and apparent brightnesses. Danger, though, of complications. See if the class can get idea of reddening, dust, or other stuff.

Bolometric luminosity versus luminosity in different bands.

Spectral types: OBAFGKMRNS. Don't worry so much about details, but general idea is that the "earlier" (towards O) the type is the hotter it is, and the "later" (towards M) the cooler it is. This earlier/later nomenclature is a relic of the idea that this actually represented an evolutionary sequence. Subdivisions within this: 0 through 9 (e.g., G2, the Sun). May also want to introduce luminosity classes: I is most luminous, VI is least luminous ("dwarfs" like the Sun are luminosity class V).

H-R diagram was a consequence of plotting observables against each other. This is often a good idea in general: if the result is a scatter plot, then no correlation, but if the result is a tight line in the plane, you may have something. This was the case in the H-R diagram.

For H-R diagram, take a good amount of time to discuss possible observational problems (for example, dust can change color) and selection effects, as well as assumptions necessary in converting from the observers' diagram (color-magnitude) to the theorists' diagram (temperature-luminosity). Bring up brightness temperature issues in, for example, pulsars.

If time: Distance determinations to globular clusters using the H-R diagram.