Ask: does everyone have experience in writing code?

Hansen & Kawaler 1.2, 1.3, 1.4

The Virial Theorem

This is a theorem relating the kinetic energy (or temperature) of particles to their potential energy.

Suppose we consider particles that are noninteracting and moving in circular orbits. What relation is there between kinetic and potential energy? Answer: for a particle of mass m in a circular orbit of radius r around a mass M, the potential energy is $-GMm/r$ and the velocity is $\sqrt{GM/r}$ so that the kinetic energy is $\frac{1}{2}GMm/r$. Therefore, $2U = -\Omega$, where U is kinetic energy and Ω is potential energy. This means that the total energy is $U + \Omega = \Omega/2 < 0$.

This already tells us something highly important about gravitation: when things lose energy, they speed up and increase in temperature. That means that gravitationally bound systems have, in this sense, negative specific heat! This is a fact of great importance to understanding such systems. This form of the theorem can be shown to apply as a time average for any system with bounded motion (Landau and Lifshitz).

Mention: globular clusters, gravothermal catastrophe.

Use this to estimate the average temperature in the Sun, assuming that the term U is related to the temperature. The velocity is $\sqrt{GM/r}$, or $v = 4.3 \times 10^7$ cm s^{-1}, so the temperature should be about $kT = \frac{1}{2}m_p v^2$, or $kT \approx 10^{-6} m_p c^2 = 10^3$ eV $= 10^7$ K! Let's do a calculation for the Earth. In that case, $v = \sqrt{GM/r} = 8 \times 10^5$ cm s^{-1}, so $kT = 4000$ K. Well, in both cases this is close to the *average* temperature.

Mention: 10^7 **K for Sun is an important result; we'll find later that something like this (**$> 10^6$ **K, say) is necessary for fusion.**

But wait: the Sun certainly isn't a bunch of noninteracting particles going in circles! If so, the *outer* portion of the Sun would be at 10^7 K, which it certainly isn't. Need a more general derivation.

State (do not derive) the virial theorem, from HK 1.3. Indicate that I will often do this, if the more detailed derivation is in the book or in my notes. **Ask class:** how can we check if the sign of the d^2I/dt^2 term is correct? Can think of just a pair of particles initially at rest. **Ask class:** how, in principle, could we check the 1/2 factor? Might think of a small particle initially at rest around a large mass, figure out change in I from standard gravitational acceleration. **Ask class:** what if there are other forces at work we're not considering? For example, skyhook holding a particle at rest. Does virial theorem still hold? No, an extra term would be needed (this adds a $-3P_sV$ on the right side; HK pg. 9). Let's think about what the "kinetic energy" term $2K$ could mean. **Ask class:** what could it mean for an orbit, for a collisionless but large stellar system, for a star, for a very cold white dwarf? Orbital velocity, velocity dispersion, average temperature, average Fermi energy.

In the derivation only gravitational forces are considered. Are we being sloppy by not including other forces? **Ask class:** what other contributions might come in? Then, lead them to realize that everything else is short-range (strong and weak are inherently short-range; EM isn't, but bulk matter is neutral for that reason). Do an order of magnitude calculation of the contribution of electrostatic potential energy.

One way: assume electrostatic force is no more than gravitational force. Then if the net charge of the Sun is Ze, the force is Ze^2/R^2, versus GMm/R^2 for gravity, so let's say $Ze^2 = GMm$. The net electrostatic potential energy is $(Ze)^2/R$ (give or take some factors), versus GM^2/R for gravity. Since $Ze = GM(m/e)$, we compare $[GM(m/e)]^2/(GM^2)$ to get $G(m/e)^2 \approx 10^{-36}$ for a proton. The electrostatic energy is negligible for objects bound by gravity.

Mention: if an astronaut removed all charges from one hair and placed them on the shuttle landing pad, the shuttle would be prevented from taking off!

Mention: this is why gravity dominates at large scales despite being by far the weakest at small scales.

Now, ask them under what conditions will the moment of inertia term play a significant role: only when the bulk velocity is a significant fraction of free-fall (if the velocity is ϵ times free-fall, the correction is of order ϵ). Do we have to worry about this? Ask for Sun, for Cepheids. Point out that again, it depends on the accuracy you want. $I = mr^2$ for single particle, $\ddot{I} = 2m\dot{r}^2 + 2mr\ddot{r}$. For radial free-fall, $\dot{r} = -\sqrt{2GM/r}$, so $\frac{1}{2}\ddot{I} = 2GM^2/R - GM^2/R = GM^2/R$, which is the magnitude of the sum of the terms on the right. Therefore, only comparable if $\dot{r}$ is comparable to free-fall.

State $\int_{\mathcal{M}} \frac{3P}{\rho} d\mathcal{M}_r + \Omega$ form, refer to H+K for derivation. However, qualitatively justify the relation with the pressure as in H+K.

Relativistic corrections? How about for the outer portion of the Sun? If there are any, they would be of order β^2. One way that we can do this is to use the first example above to indicate that the thermal velocity will be about equal to the Keplerian orbital velocity. That would be $\sqrt{GM/r}$, or 4.3×10^7 cm s^{-1}. This gives a correction of about 10^{-6}. Indicate importance of $T = 4 \times 10^9$ K for electrons, $T = 8 \times 10^{12}$ K for nucleons (have class derive both). Note that in the center of the Sun, sharing of energy between electrons and protons means $T = 1.5 \times 10^7$ K for both, so 0.3% corrections for electrons; can be substantial for more massive stars late in evolution.

Application: Kelvin-Helmholtz time scale. H+K give the details for a γ-law EOS. Refer students to the book for such details, but use instead the simple estimate of $t_{\rm KH} = GM^2/(LR)$, or about 3×10^7 yr for the Sun. Discuss history of this, how Kelvin used this as an argument against evolution (K-H time for the Earth is similarly small).

Now follow H+K 1.3.1: introduce γ-law EOS and derive total energy. Assume we have a simple relation

$$P = (\gamma - 1)\rho E \tag{1}$$

between the pressure P, the density ρ, and the internal energy per gram E. This is a "gamma-law" equation of state, and often represents reality pretty well (e.g., a monatomic ideal gas has $\gamma = 5/3$). Since $2K = 3(\gamma - 1)\int E\, dM_r$, we have $K = \frac{3}{2}(\gamma - 1)U$, where U is the internal energy. The virial theorem becomes

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 3(\gamma - 1)U + \Omega\ . \tag{2}$$

Let the total energy be $W = U + \Omega$. For something in hydrostatic equilibrium, where $d^2I/dt^2 = 0$, we then have

$$W = \frac{3\gamma - 4}{3(\gamma - 1)}\Omega\ . \tag{3}$$

Emphasize important properties of $\gamma = 4/3$. Ask class for what would happen if $W > 0$ (star could unbind itself). Foreshadow $\gamma = 4/3$ relation to relativistic EOS (e.g., radiation).

Dynamical time scale. How long would it take for the Sun to react to a sudden and dynamic change (e.g., removing pressure support?). We will have calculated this earlier: it's proportional to $1/\sqrt{G\bar{\rho}}$, or about an hour for the Sun. **Mention:** part of homework will be to work through several other such time scales and see that they are quite close to each other. Note, however, H+K result that if one is really careful the time scale is actually

$$P \approx \frac{1}{(G\bar{\rho})^{1/2}(3\Gamma_1 - 4)^{1/2}}\ , \tag{4}$$

where $\Gamma_1 = \left(\frac{d\ln P}{d\ln\rho}\right)_{\rm ad}$. Be sure to point out the role of $\Gamma_1 = 4/3$, **Ask class:** what does it mean? (It means that a star with such an equation of state will not react to a sound wave in the same way; for example, if you squeeze such a star then it stays squeezed).

Mean Molecular Weights

Before heading further, we need to define a quantity called mean molecular weight, often denoted by μ. It is simply the average mass per particle (or subset of particles), in units of a proton rest mass. For example, suppose we have only neutral atomic hydrogen.

The mean molecular weight is essentially 1, because for each particle (an atom) there is roughly one proton mass (the electron is negligibly massive). Now suppose we have fully ionized atomic hydrogen. The number of particles is now doubled, since where before we had just a single atom, now we have a proton and an electron. The mean molecular weight is then 1/2. The mean molecular weight per electron is $\mu_e = 1$, because for every electron there is one proton. As yet another example, suppose we have pure neutral helium. Here we have four proton masses (really two proton and two neutron masses, but that's close) per atom, so $\mu = 4$. If the helium were completely ionized, we'd now have three times as many particles (one nucleus plus two electrons), so $\mu = 4/3$. The mean molecular weight per electron is $\mu_e = 4/2 = 2$.

The use of this comes in different ways: for example, at a given temperature each particle contributes the same amount of pressure if it's close to an ideal gas ($P = nkT$). Thus, for a given pressure the required temperature is greater if there are fewer particles. Consider fusion inside a main sequence star. Everything is fully ionized, and four protons (plus four electrons) are converted into one helium nucleus plus two electrons. Net result: decrease of pressure, requiring that the core heat up. This explains why $T = 4.1 \times 10^6 \mu (M/M_\odot)^{2/3} \rho^{1/3}$ K in the virial temperature estimate of HK (eq. 1.33).

A Simple Model

Now let's think about a simple stellar model and see what it can tell us about stars. In fact, let's pick something stupidly simple: a star with constant density $\rho = \rho_c$(!). This is crazy, of course, but it's easy to calculate and gives some insight.

Clearly, the mass interior to some radius r is $M_r = \frac{4}{3}\pi r^3 \rho_c$. This is true all the way up to the surface $r = R$, where $M_r = M$. Then

$$M_r = \frac{r^3}{R^3} M \ . \tag{5}$$

We now insert this into the Lagrangian form of the equation of hydrostatic equilibrium:

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} = -\frac{GM}{4\pi R^4}\left(\frac{M_r}{M}\right)^{-1/3} \ . \tag{6}$$

The pressure at the surface is zero, so with this as a boundary condition we can integrate to get

$$P = P_c\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{7}$$

where $P_c = (3/8\pi)(GM^2/R^4)$ is the central pressure. This is actually a lower limit to what the pressure can be in a hydrostatic star. One can also convince oneself that GM^2/R^4 is the only combination of G, M, and R that gives units of pressure, so the central pressure always has to be proportional to that. It is straightforward to verify the virial theorem for this case. If the pressure is supplied by an ideal gas, $P = nkT$ (where, remember, $n \propto 1/\mu$), this implies a central temperature of

$$T_c = 1.15 \times 10^7 \mu (M/M_\odot)(R/R_\odot)^{-1}\,\mathrm{K}\,. \tag{8}$$

This is actually better than what we got using the simple virial estimate, because we're thinking now about the central temperature instead of some "average" temperature.