ASTR 606
Example problems

Here are a few example problems in which an inherently complicated formula can be simplified due to very large/small numbers being involved.

1. Consider a gas of atomic hydrogen for which the single-level Saha equation for pure hydrogen is valid. To within 50%, calculate the temperature at which the bound-free and free-free opacities are equal to each other for a $\hbar\omega = 100$ eV photon when n=1 cm^{-3}. The relevant cross sections are: $\sigma_{\rm bf} \approx 2 \times 10^{-14}(\hbar\omega/1\ {\rm eV})^{-3}$ cm^2 and $\sigma_{\rm ff} \approx 3 \times 10^{-35} n T^{-1/2}(\hbar\omega/1\ {\rm eV})^{-3}$ cm^2, where T is in Kelvin.

We can start by computing the ratio of the cross sections. The dependences on photon energy cancel out, so the ratio is just about $\sigma_{\rm bf}/\sigma_{\rm ff} \approx 6 \times 10^{20} T^{1/2}$. This is enormous! It means that for the opacities to be equal, there must be far more particles that can contribute to free-free opacity than to bound-free opacity. This means that the gas must be almost completely ionized. Remember that the bound-free opacity is proportional to the number of neutral atoms, which is proportional to $1 - y$, whereas the free-free opacity is proportional to the number of ionized atoms, which is proportional to y.

Armed with that knowledge, let's turn to the Saha equation:

$$\frac{y^2}{1-y} = \frac{1}{\rho} 4.01 \times 10^{-9} T^{3/2} e^{-1.578\times10^5/T} \ . \tag{1}$$

We know that $y \approx 1$, so let's write $y = 1 - \epsilon$, with $\epsilon \ll 1$. Then we have

$$\frac{1}{\epsilon} \approx \frac{1}{\rho} 4.01 \times 10^{-9} T^{3/2} e^{-1.578\times10^5/T} \ . \tag{2}$$

From the ratio of cross sections above, we know that $y/(1 - y) = 6 \times 10^{20} T^{1/2}$, or $1/\epsilon \approx 6 \times 10^{20} T^{1/2}$. Given that the mass density is $\rho = n m_H = 1.7 \times 10^{-24}$ g cm^{-3}, the Saha equation becomes

$$6 \times 10^{20} T^{1/2} = 2.4 \times 10^{15} T^{3/2} e^{-1.578\times10^5/T} \ , \tag{3}$$

or

$$T e^{-1.578\times10^5/T} = 2.5 \times 10^5 \ . \tag{4}$$

The number on the right is large. That means that the exponential is close to 1, so we could guess $T = 2.5 \times 10^5$. This is within 50% of the correct answer, which is about $T = 3.8 \times 10^5$ K.

2. (a) Suppose the center of a star has a composition $X = 0.7$, $Y = 0.28$, $Z = 0.02$. To within 50%, at what temperature is the energy generation rate by CNO burning of hydrogen equal to the energy generation rate by the p-p chain?

The rates are:

$$\epsilon_{p-p} = \frac{2.4 \times 10^4 \rho X^2}{T_9^{2/3}} \exp\left(-3.38/T_9^{1/3}\right) \tag{5}$$

and

$$\epsilon_{CNO} = \frac{4.4 \times 10^{25} \rho X Z}{T_9^{2/3}} \exp\left(-15.228/T_9^{1/3}\right) \ . \tag{6}$$

Setting the two equal to each other, we can cancel out the common factors $(\rho X/T_9^{2/3})$ to get

$$X \exp\left(-3.38/T_9^{1/3}\right) = 1.8 \times 10^{21} Z \exp\left(-15.228/T_9^{1/3}\right) \ . \tag{7}$$

The variable of interest is T, but let's define $\alpha = T_9^{-1/3}$ and take the logarithm of both sides to get

$$-3.38\alpha \approx 45 - 15.228\alpha \tag{8}$$

(where we've put in $X = 0.7$ and $Z = 0.02$), so α=3.83 and therefore $T \approx 1.8 \times 10^7$ K.

(b) That one was relatively easy because everything but the exponentials and a constant factor cancelled out. For a greater challenge: with the same composition and an assumed central density of 100 g cm^{-3}, at what temperature (to within 50%) is the energy generation rate from the triple-alpha process equal to that from p-p burning?

The triple-alpha rate is

$$\epsilon_{3\alpha} = \frac{5.1 \times 10^8 \rho^2 Y^3}{T_9^3} \exp\left(-4.4/T_9\right) \ . \tag{9}$$

Setting this equal to the p-p rate and cancelling some common factors,

$$\exp\left(-3.38/T_9^{1/3}\right) \approx 10^3 \rho T_9^{-7/3} \exp\left(-4.4/T_9\right) \ , \tag{10}$$

where in this expression we substituted $X = 0.7$ and $Y = 0.28$. How do we solve this? We know that the triple-alpha reaction is very temperature-sensitive, so as a first crack we can assume that the exponentials dominate completely. We will therefore simplify to

$$\exp\left(-3.38/T_9^{1/3}\right) \approx 10^3 \exp\left(-4.4/T_9\right) \tag{11}$$

and take the log. As before, let $\alpha = T_9^{-1/3}$, so we have

$$-3.38\alpha \approx 7 - 4.4\alpha^3 \ . \tag{12}$$

Cubic equations have an analytic solution, but it's nasty. We know, though, that $\alpha > 1$ (since $T_9 < 1$) but not by an order of magnitude (since certainly $T > 10^7$ K). Let's substitute

$\alpha = 1$ in the left side; the justification is that because we will then take a cube root, our answer won't depend much on our assumption. We find $\alpha \approx (10.38/4.4)^{1/3} = 1.33$. Note that if we were to put this back in the equation and solve $\alpha = [(7 + 3.38 * 1.33)/4.4]^{1/3}$, we would get 1.38. There is practically no difference.

So, we're getting $T = 10^9\alpha^{-3} \approx 4 \times 10^8$ K. But you remember that as our initial simplification step we dropped a factor of $T_9^{-7/3}$. Did this affect our answer much? We can now put in $T_9 = 0.4$ and see what happens. This means that the constant factor is now $10^3(0.4)^{-7/3} = 8500$, so the logarithm is 9 instead of 7. Our answer then becomes $\alpha = 1.41$, so $T = 10^9\alpha^{-3} \approx 3.6 \times 10^8$ K. The answer is virtually identical.

I'm not going to ask you anything nearly this complicated. But it does demonstrate a couple of ways in which one can simplify an equation. It also shows that if you're in doubt about an answer, you can always iterate (i.e., put your first guess back in the equation, and solve again). If the answer doesn't change much, it's reliable.

3. Here's an example from high energy astrophysics. A photon with sufficient energy can produce an electron-positron pair by some different processes, including scattering off another photon ($\gamma\gamma \to e^-e^+$) and single-photon pair production off of a magnetic field ($\gamma B \to e^-e^+B$). The cross section for photon-photon pair production is, for $(\hbar\omega)^2 \gg (m_ec^2)^2$,

$$\sigma_{\gamma\gamma} \approx \frac{3}{8}\sigma_T \left(\frac{mc^2}{\hbar\omega}\right)^2 \left(\ln\frac{2\hbar\omega}{mc^2} - 1\right) . \tag{13}$$

Here σ_T is the Thomson cross section. For single-photon pair production the photon energy must be $\hbar\omega > 2m_ec^2$. The mean free path for this process d_0, such that the number of photons is attenuated by e^{-d/d_0} in a distance d, is

$$d_0 \approx 2 \times 10^{-8} \left(\frac{4.4 \times 10^{13}\,\mathrm{G}}{B}\right) \exp\left[\left(\frac{6 \times 10^{13}\,\mathrm{G}}{B}\right)\left(\frac{m_ec^2}{\hbar\omega}\right)\right] \mathrm{cm} . \tag{14}$$

Suppose $\hbar\omega = 100m_ec^2$, and the number density of photons is $n = 10^{20}$ cm^{-3}. Within a factor of two, what is the lower limit to the magnetic field such that the mean free path to single-photon pair production is less than the mean free path to photon-photon pair production?

Looks complicated, doesn't it? As usual, though, we have lots of simplifications we can use. At $\hbar\omega = 100m_ec^2$, the photon-photon pair production cross section is $\sigma_{\gamma\gamma} \approx 10^{-4}\sigma_T$. The mean free path is $\ell_{\gamma\gamma} = 1/(n\sigma) \approx 1.5 \times 10^8$ cm. Setting $d_0 = 1.5 \times 10^8$ cm gives us

$$\left(\frac{4.4 \times 10^{13}\,\mathrm{G}}{B}\right) \exp\left[\left(\frac{6 \times 10^{13}\,\mathrm{G}}{B}\right)\left(\frac{m_ec^2}{\hbar\omega}\right)\right] \approx 7 \times 10^{15} . \tag{15}$$

With such a large answer, the factor out front is essentially irrelevant. Substituting $\hbar\omega = 100 m_e c^2$, we have

$$\exp\left[0.01\left(\frac{6\times 10^{13}\ \mathrm{G}}{B}\right)\right] \approx 7\times 10^{15}\ . \tag{16}$$

Solving, $B \approx 1.6\times 10^{10}$ G. If we were to put this back in the original equation and solve again, we'd get $B \approx 2.1\times 10^{10}$ G, so our first answer was good enough to the required factor of 2.