ASTR 601 Problem Set 2: Due Thursday, October 2

1. (4 points) Another application of detailed balance is in probability distributions. Suppose that we have a probability density $p(\vec{x})$, where $\vec{x}$ is a vector of parameters in a model. The probability density is normalized so that $\int p(\vec{x})d\vec{x} = 1$. Suppose that we are exploring this distribution numerically, and that we start with a particular vector $\vec{x}_1$ of parameters. As a step in our exploration, there is some probability that we will move to another vector $\vec{x}_2$ of parameters. Let $q(\vec{x}_2|\vec{x}_1)$ be the probability of accepting this step (i.e., actually taking it) and similarly let $q(\vec{x}_1|\vec{x}_2)$ be the probability that a proposed step from $\vec{x}_2$ to $\vec{x}_1$ is accepted. As usual with probabilities, $q = 1$ would mean that the step, if proposed, is certain to be taken and $q = 0$ would mean that the step is guaranteed not to be taken. Derive values for $q(\vec{x}_2|\vec{x}_1)$ and $q(\vec{x}_1|\vec{x}_2)$, which depend on their relative probability densities $p(\vec{x}_1)$ and $p(\vec{x}_2)$, such that the probability distribution is maintained after many such candidate steps. This set of q values would then satisfy detailed balance.

2. (4 points) The average temperature of the cosmic microwave background right now is about 2.7 K, and the CMB is nearly a perfect blackbody. Let us suppose that the average temperature of a star in the Milky Way is 6,000 K (we will assume that stars emit as isotropic blackbodies) and that the total stellar luminosity of the Milky Way is 1.5×10^{44} erg s^{-1}. Let us also assume that the Milky Way is a sphere with radius 300 kpc (i.e., out to the dark matter radius), and that all of the starlight is produced at the central point of the Milky Way (because the luminous part of the Milky Way is much smaller than the dark matter radius). Given these assumptions, calculate the ratio between the number of CMB photons and the number of stellar photons in the Milky Way at any given instant. Recall that the energy density of blackbody radiation of temperature T is $U = aT^4$, where $a = 7.57 \times 10^{-15}$ erg cm^{-3} K^{-4}. **Hint:** do stars fill the entire 300 kpc radius volume? How about the CMB?

3. (4 points) Write a code to compute, and plot, the ionization fraction y over a range of temperatures from $T = 10^3$ K to $T = 10^6$ K, in logarithmic steps $d\log_{10} T(\mathrm{K}) = 1$ for $\rho = 10^{-31}$ g cm^{-3} (roughly the average baryon density of the universe), $\rho = 10^{-24}$ g cm^{-3} (representative of the average density of the interstellar medium), and $\rho = 10^{-16}$ g cm^{-3} (a reasonable density for a core of a molecular cloud). What trends do you see, and how would you explain them? **Please send me a copy of your code before you hand in the plots on the due date. Any language is fine as long as it compiles and runs on my departmental machine (please send me compilation/run instructions); I won't install any libraries or download modules!** Please ensure that when your code runs, it produces both a plot of $\log_{10} y$ versus $\log_{10} \rho$ for each temperature (which is all you

need for the hardcopy) and a table of $\log_{10} y$ versus $\log_{10} \rho$ for each temperature. In the table, your values of $\log_{10} y$ must be correct to at least three significant figures. To do that, please use the high-precision version of the Saha equation

$$\frac{y^2}{1-y} = \frac{4.0355 \times 10^{-9}}{\rho} T^{3/2} e^{-1.57887\times10^5/T} \ , \tag{1}$$

where ρ is the *total* mass density (including protons) in g cm^{-3} and T is the temperature in K. **For this problem set only, if you get drafts of your codes to me well in advance then I will give you hints about whether your numbers are accurate. From this point on, however, I will simply acknowledge receipt of the code and indicate whether it ran, if I get it early enough before class.**

4. (4 points) Michael Faraday introduced a valuable way to think intuitively about magnetic fields: you consider magnetic field lines to be almost physical entities, which can wind, stretch, and move around but not break (because breaking would imply magnetic monopoles). With this in mind, Dr. Sane has realized that a terrible fate awaits our Galaxy... unless we act now! You see, the center of our Galaxy has magnetic fields (from past episodes of star formation, among other things). It also has a central supermassive black hole (SMBH), and other (secondary) black holes that spiral in from time to time. The problem is that when a secondary black hole spirals toward the SMBH, it winds up magnetic fields. Because magnetic energy density scales as B^2, this rapidly increases the energy in magnetic fields, with horrible consequences (giant bursts of radiation, rapid decays of orbits, dogs and cats living together...). He has sent out a press release to this effect, and the New York Times has called you for your perspective.

Focusing on one particular issue, the Times would like you to explain the following. We can imagine magnetic field lines being twisted and amplified in the plane of the orbit of the secondary black hole. Let's even simplify: we will ignore all other stars, gas, etc., and focus just on this single orbit. Please discuss whether the windup and amplification will continue to increase the magnetic field, or whether there is a way to release much of the magnetic tension. The way we have in mind is nothing complicated (e.g., you don't need to know anything about magnetohydrodynamics, or kinking of the field lines, or reconnection!) **Hint:** think in three dimensions.