ASTR 320
Problem Set 5
Due Thursday, May 1

1. Suppose that you have two fluids in a large container. One fluid (call it fluid 2) is on top of the other (call it fluid 1); the density of fluid 2 is ρ_2 and the density of fluid 1 is ρ_1. The boundary between the two is a horizontal plane, and the whole system is in a uniform gravitational field of acceleration g, pointed down (i.e., vertically). The system is initially in equilibrium.

(a) Given an argument based on potential energy that shows that this setup will be unstable if $\rho_2 > \rho_1$. **Hint:** what happens if you switch a fluid element of fluid 2 with a fluid element of fluid 1, near the boundary?

(b) Let the frequency of oscillation be ω (units: s^{-1}) for a perturbation of wavenumber k (units: cm^{-1}). Write an approximate formula for ω^2 in terms of g, k, ρ_1, and ρ_2. **Hint:** get the general idea by writing an equation that has the right units, then make sure that your expression gives reasonable answers in limits such as $\rho_2 > \rho_1$, $\rho_2 = \rho_1$, and $\rho_2 < \rho_1$.

2. A muon is an elementary particle with the same electric charge as an electron, but roughly 200 times the mass of an electron. We will consider "muonium", which is like hydrogen except that there is a single muon orbiting the proton instead of a single electron orbiting the proton. In this problem, we will ignore the fact that the proton has finite mass (i.e., you should use the muon mass m_μ instead of the reduced mass).

(a) Using the uncertainty principle as in class, estimate the energy and typical size of muonium in its ground state.

(b) Suppose that the nucleus has charge Ze instead of e. Following the derivation in the class notes, solve the Schrödinger equation to get the ground state energy for muonium. Within a factor of 2, estimate the value of Z at which the ground state energy is $\sim$10% of the rest mass energy of the muon. This would mean that relativistic effects are significant, so the Schrödinger equation is not so accurate.

3. In free space, neutrons are unstable because their mass is larger than the mass of a proton plus the mass of an electron. However, at high densities, the electron Fermi energy is large enough that this is no longer true, and instead there is a tendency for protons and electrons to squeeze together to form neutrons. Thus, neutron stars are primarily neutrons (instead of being like regular matter).

The electron Fermi energy becomes equal to its rest mass energy at a density of roughly $10^6\ \mathrm{g\ cm^{-3}}$. The rest mass of an electron is $m_e = 9.11 \times 10^{-28}$ g, of a proton is

$m_p = 1.6726 \times 10^{-24}$ g, and of a neutron is $m_n = 1.6749 \times 10^{-24}$ g. To within a factor of 2, what is the density at which the total energy of an electron (that is, its rest mass energy plus its Fermi energy) equals the mass-energy difference between a neutron and a proton, and hence neutrons start to become more common? **Hint:** is the Fermi energy in the nonrelativistic or relativistic domain? This will tell you how to extrapolate the Fermi energy with density.

4. Dr. Sane has decided to work a bit on observations. He has looked at a molecular cloud, and has observed what he believes to be strong rotational transitions of the H_2 molecule. He also thinks that the temperature of the cloud is $T = 10$ K. Convince him that he is making a mistake one way or the other:

(a) First, consider a molecule of moment of inertia I and rotational frequency Ω. From quantum principles, its angular momentum must be $n\hbar$, where $n \geq 1$ is an integer. Determine the minimum energy of a rotational transition of this molecule (i.e., from one angular momentum to another).

(b) The moment of inertia is roughly $I \sim MR^2$, where M is the mass of the molecule and R is its radius. If R is roughly a Bohr radius, estimate the minimum temperature (to within a factor of 3) at which you would expect to be able to see a rotational transition of H_2. **Hint:** the thermal energy has to be at least comparable to the energy of the transition. Boltzmann's constant is $k = 1.38 \times 10^{-16}$ erg K^{-1}.