ASTR 320
Problem Set 3
Due Thursday, March 13

1. Suppose you have a globular cluster with mass $10^5\,M_\odot$ and radius 10 pc, that is spherically symmetric, time-independent, and has uniform density. Two $10^3\,M_\odot$ black holes in this cluster start 1 pc apart and tighten by a series of interactions with stars to a separation of 1 AU. From that point they interact with no more stars, but gravitational radiation causes the black holes to merge with each other. No interaction with a star throws that star out of the cluster (i.e., all are retained). Assume that afterwards, the cluster is a uniform sphere but has a new radius. To within a factor of two, what is that new radius?

2. Dr. I. M. N. Sane has written you with an exciting new theoretical discovery. He believes that there is a certain class of stars, which he modestly calls "Sane stars", that undergo large oscillations. He believes that the stars have constant density as a function of radius, but that the density changes so that the whole radius of the star expands and contracts by 10% in a cycle:

$$R = R_0[1 + 0.1\sin(\omega t)]\ . \tag{1}$$

The mass of such a star is $M = 10\,M_\odot$, the radius is $R = 10\,R_\odot$, and the period of oscillation is $P = 2\pi/\omega$ =10 days. The reason Dr. Sane is so excited is that he thinks virial equilibrium will be violated wildly, in the sense that the $\frac{1}{2}d^2I/dt^2$ term will be comparable to $W + 2K$, where W is the potential energy and K is the kinetic energy. Evaluate his claim by estimating, to within a factor of 10, the magnitudes of $W + 2K$ and $\frac{1}{2}d^2I/dt^2$ for Sane stars.

3. Suppose that a mass distribution is time-independent and spherically symmetric and has the property that a particle in a circular orbit a distance r from the center has an orbital velocity $v_0 > 0$ that is independent of r.

(a) Compute the density and potential for this distribution as a function of radius.

(b) Calculate the density at $r = 0$, and the total mass integrated from $r = 0$ to $r = \infty$. Qualitatively, explain how the density distribution must be modified at very small and very large radii to make it physically realistic.

4. Demonstrate explicitly that for a uniform-density spherically symmetric collection of matter of mass M and radius R, the total gravitational potential energy is $W = -\frac{3}{5}GM^2/R$. Assume time-independence.