ASTR 320
Problem Set 2
Due Thursday, February 27

1. Practice derivations. When we did the two-body problem we defined

$$\mathbf{r} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} , \tag{1}$$

$\mathbf{R} \equiv \mathbf{r}_1 - \mathbf{r}_2$, $M_{\rm tot} = m_1 + m_2$, and $\mu = m_1 m_2 / M_{\rm tot}$, where m_1 and m_2 are the two masses and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the masses in some inertial coordinate system.

(a) Show that the total energy,

$$E = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|} , \tag{2}$$

can be written in the form

$$E = \frac{1}{2}M_{\rm tot}|\dot{\mathbf{r}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{R}}|^2 - \frac{G\mu M_{\rm tot}}{|\mathbf{R}|} . \tag{3}$$

(b) Show that the total angular momentum,

$$\mathbf{L} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 , \tag{4}$$

can be written in the form

$$\mathbf{L} = M_{\rm tot}\mathbf{r} \times \dot{\mathbf{r}} + \mu\mathbf{R} \times \dot{\mathbf{R}} . \tag{5}$$

2. Black holes in globular clusters are an exciting topic of current astrophysics. One question relates to whether interactions between single black holes and binary black holes can lead to mergers that may be detectable with gravitational wave instruments. Consider two $10\,M_\odot$ black holes in a binary of semimajor axis a, interacting with a single $10\,M_\odot$ black hole that comes from a large distance with an initial speed of essentially zero. Such a three-body interaction will tighten the binary; typically, the semimajor axis a will decrease by roughly 20%. This will cause the binary (and single black hole) to recoil. A black hole binary of this type must reach $a = 10^{12}$ cm to merge by gravitational radiation.

(a) Calculate the recoil speed of the binary as a function of a, assuming a change of 20% in $1/a$ (not a itself; the change in $1/a$ makes the algebra easier). Let $M = 10\,M_\odot$ be the mass of each of the three black holes.

(b) The escape velocity from the center of a globular cluster is about 50 km s^{-1}, whereas the escape velocity from the center of a large galaxy can be 500 km s^{-1}. Assume the binary black holes start at a large separation and tighten by repeated encounters with single black holes. Will the binary black holes merge first (i.e., tighten to a separation of 10^{12} cm) or be kicked out first in a globular? What about in the center of a galaxy?

Hint: use conservation laws for this problem, and work in the center of mass frame between the binary and single black holes!

3. Dr. I. M. N. Sane, daring theorist extraordinaire, has been interested in different generations of stars: Population I, II, and III stars. He proposes "Pop IV" stars that are currently forming from the intergalactic medium. He thinks that these stars typically have a mass of 1 $M_\odot$. Leslie Sage, astronomy editor of *Nature*, has asked you to review this paper. Leslie tells you that the intergalactic medium has an average density of $\rho = 10^{-31}$ g cm^{-3} and that hydrogen molecules in the IGM typically move at a speed $v \approx 100$ km s^{-1}. Leslie also says that the minimum mass that can form from a medium is determined by the mass of the smallest sphere that is self-bound, meaning that the total energy (kinetic plus potential) of matter in it is negative. Assuming only gravitational potential energy matters, estimate this minimum mass to within an order of magnitude and hence evaluate Dr. Sane's idea.

4. In the far future, a remarkable double planet has been discovered, where the two masses are almost equal: the first planet has a mass of m_1 and the second has a mass $m_2 = m_1(1 + \epsilon)$, where $|\epsilon| \ll 1$. They orbit each other in a circle of semimajor axis a. The local government wants to put a space station at the L_1 point, between the two planets. They have consulted you as to roughly where they should put the station. Your task is to determine the location of the L_1 point to lowest nonzero order in ϵ. Assume that in the rotating frame, the two planets and the station are all on the x axis, that planet 1 is at $x < 0$, that planet 2 is at $x > 0$, and that the center of mass is at $x = 0$. Check the units, limits, and symmetries of your answer!

Hint: assume that the L_1 point is at a location x, where $|x| \ll a$. When you do your expansion, keep terms of order x and ϵ, but drop higher-order terms, such as x^2, ϵ^2, and $x\epsilon$.