Equilibrium

When last we left off, we had discussed thermodynamics and statistical mechanics. From these, we can distill two important principles. One is conservation of energy, which can be expressed in the following way for a gas with pressure P, temperature T, and chemical potential μ:

$$dU = -PdV + TdS + \mu dN \ , \tag{1}$$

where dV is the change in volume, dS is the change in entropy, and dN is the change in the number of particles. Note that this assumes that, prior to the changes indicated by the differentials (i.e., prior to changing the volume, entropy, or number of particles), the gas was in equilibrium. Clearly, the internal energy of a gas doesn't change if you just let it sit in an isolated container, but the formula above would predict that it does if the gas velocities were initially ordered, because the entropy would change. Therefore, only if the gas is initially in equilibrium does this apply. For that matter, strictly speaking, assigning a single temperature T to a gas out of equilibrium is troubling!

The second principle is that, again in equilibrium, the classical formula for the population of a state (including position, velocities, and internal energies) is

$$N(\text{state}) \propto \exp(-E_{\text{tot}}/kT) \ . \tag{2}$$

For both these results, it is obviously crucial that the systems be in equilibrium. Indeed, there are many situations in astrophysics when one makes the assumption of equilibrium. We now need to delve deeper into what this means.

Ask class: what does it mean that a system is in equilibrium? This is not a trivial question! One try might be to say that a system is in equilibrium if, in some time-averaged sense, the properties of the system are time-independent. Or, you could say that a system is in equilibrium if over a long haul all processes and their reverses are in balance. There are, however, many different forms of equilibrium, so we should think about some specific examples. The point of all of this is that analyses are simplified dramatically when equilibrium can be assumed, so we'll save ourselves a lot of effort if we can establish that a system is in equilibrium.

Let's try an example. **Ask class:** is a rock on a table in equilibrium? More precisely, we can ask the ways in which it can be in equilibrium, then determine if it is. **Ask class:** what are specific ways in which the rock could be in equilibrium? One is the question of whether it is moving. It isn't. Another would be whether it is at a constant temperature. It may or may not be, but we know that if we wait long enough it will acquire room temperature, then it won't change its temperature. A third type of equilibrium is chemical. If, for example, the rock has a lot of iron in it, then we know if we wait for a very long time that iron will rust, so in fact the rock is not in chemical equilibrium until that happens. Yet another type of equilibrium is nuclear. It happens that at low pressures (such as in everyday life!), the atomic nucleus with the greatest amount of binding energy per nucleon is ^{56}Fe. Nuclei lighter than ^{56}Fe can fuse to release energy, and heavier nuclei can split apart to release energy. A typical rock is certainly not in nuclear equilibrium. It contains silicon and oxygen, which are lighter than iron. In principle, if we were to wait for unfathomable amounts of time (much more than a googol years!) then the rock's nuclei would gradually fuse with each other to form iron, by quantum tunneling, but this ain't gonna happen any time soon.

Ask class: is the Sun in equilibrium? It is again helpful to break this down in terms of different processes that might balance or be time-dependent. For example, **Ask class:** is the Sun collapsing or exploding? No, of course! That means it isn't dramatically out of hydrostatic equilibrium, which is the balance between gravity and pressure gradients (and we'll treat it in more detail later in this lecture). **Ask class:** is the Sun completely still? No, there are prominences and flares, and the Sun also vibrates with millions of modes. Therefore, the Sun is not far out of hydrostatic equilibrium, but it isn't in perfect equilibrium either. **Ask class:** what about temperature equilibrium? The Sun is about 6,000 K at the surface, radiating into a 3 K blackbody, and is about 1.5×10^7 K in its center. These temperature differences mean that energy has a one-way flow from hot to cold, so this is clearly an nonequilibrium situation. **Ask class:** what about nuclear equilibrium? Nope, the Sun fails that as well, since it isn't made primarily of iron!

Our net conclusion is that the Sun is absolutely not in equilibrium. However, we also know that the Sun isn't far from equilibrium in some ways. For example, its temperature doesn't randomly change by a factor of two over a few seconds! Thus, for some purposes, we can assume that it is close to equilibrium, and the error is small. This leads to a question that one should always ask (at least implicitly) when trying to model an astrophysical system:

What fractional error is introduced by assuming different types of equilibrium?

For example, suppose we assume that for the purposes of figuring out how energy is transported in the Sun, the Sun is in thermal equilibrium in small regions. One way to evaluate this is to ask how great a temperature change is sampled by a typical photon between two successive interactions. In most places in the Sun, a photon goes less than 1 cm before it scatters or is absorbed. If you assume a constant temperature gradient of about 10^7 K over the $\sim 10^{11}$ cm radius of the Sun, this suggests that the temperature changes by only 10^{-4} K in that distance. This is much less than even the 6,000 K surface temperature of the Sun, so the change in temperature is tiny. Thus, to an awfully good approximation, the interior of the Sun is in thermal equilibrium. This means that the laws of thermodynamics work well. However, if you want to know how energy is transported from the hot interior of the Sun to the cold exterior, it is precisely that temperature gradient that matters, so you can't ignore it. If you ask about the corona or chromosphere of the Sun, photons can travel freely in them (that's why we see the deeper photosphere), so the changes in temperature sampled by photons are huge.

These are merely specific examples of a much more general principle of astrophysical modeling. Remember: *all* equations you're likely to use contain implicit assumptions. You need them to make progress. You therefore need to strike a balance between simplifications (required so things are simple enough to solve analytically or on a computer) and realism. Therefore, any time you use an equation or a physical picture, you must know its domain of applicability and the likely errors you will make by not fully solving the Dirac equation for everything :). Folded in with this is the fact that measurements of objects are imprecise and incomplete, and if those uncertainties are greater than the error you make in using a simplification, then going beyond that simplification is pointless. Astrophysical intuition is developed gradually, through experience, but some of it consists of calculations of when you are justified in ignoring certain complications. Eventually, you do enough of these calculations that when a similar situation arises you don't have to do the same checks from scratch.

With that in mind, we will now examine a particular assumption, that of hydrostatic equilibrium, that is a starting point for the modeling of many astronomical objects.

Dynamic, or hydrostatic, equilibrium.—This means that the object as a whole stays put. Said another way, the forces acting on any given parcel of gas balance each other. First **Ask class:** what would happen if the Sun were far away from this balance? Answer: it would collapse or expand on the dynamic time scale, which to a decent approximation is just the free-fall time scale. That, in turn, is roughly $1/\sqrt{GM/R^3}$, or about $1/\sqrt{G\rho}$. For the Sun, the average density ρ is about 1 g cm^{-3}, so that's 1 hour. Since we don't see dramatic changes in the Sun on 1 hour time scales, and indeed not on scales of millions of years (**Ask class:** how do we know? Fossil, geologic record.), we know that this overall balance holds to extreme accuracy. Not a good approximation for supernovae, of course, but even for most pulsators the bulk of the star is in hydrostatic equilibrium.

Now need to quantify what is balancing what. For a given parcel of gas, gravity pulls down. **Ask class:** what could oppose gravity? If the pressure *gradient* (not just the pressure) is in the same direction of gravity (i.e., more pressure farther down), then this opposes gravity. Let's say that we have a parcel of gas with area perpendicular to $\hat{r}$ of A and thickness dr. If the density is ρ, then the gravitational force on this is $-GM\rho Adr/r^2$, where the negative sign indicates a downward force. This can be written as $g\rho Adr$, where $g = -GM/r^2$ should really be a vector, and of course M is really M_r, the mass interior to r. The force due to the pressure gradient is $P(r)A - P(r+dr)A = dr(dP/dr)A$. The sum of the two has to be zero for force balance, so $dr(dP/dr)A + g\rho Adr = 0$, or finally

$dP/dr = -\rho g$ (more generally, when spherical symmetry doesn't apply, $\nabla P = \rho \mathbf{g}$; in the previous case, $\mathbf{g} = -g\hat{\mathbf{r}}$). **Ask class:** does this have the right units, limits? This is the equation of hydrostatic equilibrium, and is one of the four fundamental equations of stellar structure. **Ask class:** how would this be modified if a star were rotating rapidly? Have to include centrifugal terms. In general, need force balance for dynamic equilibrium. In the case of stars (and most other things in the universe), this translates to gravity vs. everything else, because gravity is universally attractive and hence other forces are needed to balance it.

It is often helpful to write such equations in terms of the mass instead of the radius. This formulation, in which we follow the mass, is called the *Lagrangian* formulation. Then, since the mass in a spherical shell is $dM_r = 4\pi r^2 \rho dr$, the equation of hydrostatic equilibrium is $dP/dM_r = -GM_r/4\pi r^4$.

In the next several classes we will examine a number of aspects of gas physics and fluid dynamics. It is helpful to remember that many equations in fluid mechanics are really expressions of a single master equation, that many of you may have seen:

$$F = ma\ . \tag{3}$$

Yep! It's just Newton's law. At first glance, the particular equation may look a lot different from this. For example, the Navier-Stokes equation (which we will *not*, repeat *not* use in this class!) is

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla P + \eta\nabla^2\mathbf{v} + (\zeta + \frac{1}{3}\eta)\nabla(\nabla\cdot\mathbf{v})\ . \tag{4}$$

This looks thoroughly nasty and not at all related to $F = ma$! However, that's misleading. If you wanted to figure out the acceleration of a parcel of gas, you would of course want to add up all of contributions to the force. This might include pressure gradients, divergences in the velocity field, the effects of viscosity, and so on, and that's what all the terms on the right are doing. You also have a choice about whether to evaluate $F = ma$ in a frame moving with the fluid (the Lagrangian formulation) or in a frame fixed in a coordinate system (the Eulerian formulation). A key in astrophysics is being able to decipher the physical meaning of equations, rather than just memorizing them or running away in terror :), and in this case the equation just says that fluids move based on the forces on them.

Anyway, let's now consider an applications of hydrostatic equilibrium. First, let's think of the Earth's atmosphere. The air has a nonzero temperature, so it has a pressure. If the air is roughly an ideal gas (our first approximation) then its pressure is $P = nkT$. Let's assume that the temperature is constant (our next approximation). We'd like to know how the density of the atmosphere varies with height. We therefore ignore variations in other than the vertical direction (our third approximation). In the vertical direction, the equation of hydrostatic equilibrium becomes

$$dP/dr = -\rho g(r)\ . \tag{5}$$

Here $\rho = n\langle m\rangle$, where $\langle m\rangle$ is the average mass of an air molecule. **Ask class:** what reasonable approximation can we make about $g(r)$? We can assume it is constant. That's because there is little mass in Earth's atmosphere, and because we are most interested in heights much smaller than the radius of the Earth. Then we have

$$\begin{aligned} d(nkT)/dr &= -n\langle m\rangle g \\ (1/n)dn/dr &= -\langle m\rangle g/kT \\ n &\propto \exp(-\langle m\rangle gh/kT)\ , \end{aligned} \tag{6}$$

where h is the distance above the reference height. If we put in $T = 300$ K and $\langle m\rangle = 5\times10^{-23}$ g, roughly the mass of nitrogen or oxygen, we find that the density drops to 1/e of its maximum in a height $h \approx 8$ km (this is called a *scale height*), which is about right. Our approximations were justified. If we tried to extend this too far, however, we would find that the temperature is not constant because of solar particle heating of the upper atmosphere. Still, it's not bad.

Ask class: suppose we were to apply this analysis to the scale height of hydrogen on the Moon. Would we have to adjust it in any way? Yes, because the Moon has 1/6 of the surface gravity of the Earth, and hydrogen has a mass 1/16 of oxygen, so the scale height would be roughly 800 km, too close to the 2000 km radius of the Moon for the assumption of constant gravity to be correct.