General three-body interactions

We've been dancing around the general three-body problem, but let's imagine we've bitten the bullet and decided to do it for real. Three objects, arbitrary masses, arbitrary positions, arbitrary velocities. What happens? A nasty mess is what happens. Oh, well, we can just give it to a computer and solve, right? Wrong! Sure, we know the forces and therefore the accelerations of each object, so given the initial positions and velocities we have all the information we need in principle. In practice, however, general, close, interactions of three comparable mass objects give you chaos. This has a precise definition: a system is chaotic if slightly different initial conditions lead to behavior that deviates exponentially with time.

Let's clarify this by first thinking of a system that is *not* chaotic. Consider circular orbits around a large mass M. Think of two dust particles, one of which orbits at radius r_1 with angular frequency Ω_1, and the other of which orbits at $r_2 \approx r_1$ with angular frequency $\Omega_2 \approx \Omega_1$. Start them at the same angle ϕ_0. Then, as a function of time t, the first particle has an angle $\phi_1 = \phi_0 + \Omega_1 t$, and the second has an angle $\phi_2 = \phi_0 + \Omega_2 t$. The difference in angles is $\phi_2 - \phi_1 = (\Omega_2 - \Omega_1)t$, which therefore grows linearly with time.

In contrast, the weather is a classic system that is chaotic. Suppose you take your best measurements of the current state of the weather in your area (e.g., temperature, pressure, humidity, etc.). You plug this into your weather model to predict the temperature at noon in the next week. Various errors in measurement mean that, say, after one day the temperature may be 1°C different than you predicted. After two days, it could be 2°C different. After three days, 4°C; after four days, 8°C. Generally, your uncertainty in temperature could be written $\Delta T = 2^{t/1\text{ day}}$ °C. Therefore, the initial conditions deviate exponentially with time.

This is the case with a closely interacting three-body system. It means that, no matter how careful you are in specifying initial conditions, on the timescale of exponential growth of deviations (sometimes called the Lyapunov timescale) your accuracy will degrade quickly. Therefore, in a practical sense, even a computer can't do a perfectly accurate three-body integration forever.

Time to give up? No! Even if the three-body problem of interest is chaotic (and many aren't), being chaotic emphatically does *not* mean that any state is accessible. Consider again the problem of the weather. If you continued the exponential uncertainty, it would imply that within two weeks the uncertainty would be more than 4,000°C!! Of course there are bounds to what the system can do. In addition, there are some long-term trends that exist in chaotic systems. For example, although you may not be able to predict the temperature of a given day in June in Washington, you are safe in assuming that in the overwhelming majority of cases it will be warmer than a given day in January in Washington. Seasons are much easier to predict than specific days. Similarly, in the three-body problem, one can use conservation of energy, linear momentum, and angular momentum to place bounds on possible outcomes. In addition, even if a computer simulation deviates from the "true" result, the trends visible in many simulations of a given interaction are likely to be valid.

With that in mind, let's think a bit about stability. Suppose you have two stars of equal mass orbiting each other at a constant separation of 1 AU. **Ask class:** what will be the motion of a small planet that initially is moving in a circle of radius 10 AU centered on the center of mass? It is only slightly changed from the orbit it would have around a single object, because at that distance the force from the two stars is close to that from one star with the combined masses. **Ask class:** what happens as the initial orbital radius is decreased? Then the deviations from a $1/r^2$ law become more and more pronounced, so perturbations of the orbit become more serious. When the orbit comes too close (typically a few times the semimajor axis of the stars, where a few is ~ 3), then after just a few orbits the planet is ejected. **Ask class:** what if the planet is ten times closer to one star than to the other? Then it orbits primarily around one star, so again the orbit is relatively stable. Think, for example, of the orbit of the Moon around the Earth. The Sun applies a torque, but the orbit is nicely stable anyway because the Moon is well within the Earth's Hill sphere.

Now let's go to something a bit more dramatic, which I've been thinking about in the last couple of years. Suppose that you have a binary (composed of stars B1 and B2), and a single star (S) comes along and interacts with the binary. Let's treat all three stars as point particles for the moment. There will be some outcome of the interaction. Knowing nothing further for the moment, **Ask class:** what are possible results of the interaction, assuming Newtonian gravity? This is an issue that can be addressed using conservation laws. If we have a general interaction of this sort, we still have a limited set of outcomes: (1) the binary is disrupted, and all three stars move along as single stars, or (2) after the interaction, S still is a single star and B1 and B2 still form a binary, or (3) S and B1 are in a binary and B2 moves on as a single, or (4) S and B2 are in a binary and B1 moves on as a single. You might think that possibly (5) all three stars end up in a triple system, but in fact a triple system that is stable indefinitely is not possible in a single-binary encounter. **Ask class:** why might this be? It's not easy to think of, but the reason is that the paths are reversible. Therefore, since originally it was a binary and an unbound single star, eventually it will return to that state (or to a state of three isolated stars). This isn't obvious, but it's true and worth thinking about. If there is some way to dissipate energy, such as in tides or in gravitational radiation, then one can in principle get a stable hierarchical three-body system out of this interaction.

To make more progress in understanding binary interactions, let's back up and think about what happens when two (pointlike!) objects of equal mass interact gravitationally with each other as they pass by. Suppose we have two stars that are *not* bound to each other (i.e., they are hyperbolic). As measured in, e.g., the center of mass frame of a globular cluster, let the stars both initially travel in the $\hat{x}$ direction. That means that the center of mass between the two stars also travels in the $\hat{x}$ direction: $\mathbf{v}_{\rm CM} = v_{\rm CM}\hat{x}$. In this center of mass frame, since by assumption the stars have equal mass they must have equal speed: $\mathbf{v}_1 = v_0\hat{x}$ and $\mathbf{v}_2 = -v_0\hat{x}$. In the CM frame, the net result of the interaction will be that both stars are deflected by an angle θ. **Ask class:** why is the deflection angle the same for both? Because otherwise linear momentum would not be conserved in the CM frame. In this frame, each star still has a total speed v_0. **Ask class:** why? This comes from conservation of energy. Therefore, after the interaction, the new velocities in the center of mass frame are

$$\begin{aligned}
\mathbf{v}_{1,\rm CM} &= v_0\cos\theta\hat{x} + v_0\sin\theta\hat{y} \\
\mathbf{v}_{2,\rm CM} &= -v_0\cos\theta\hat{x} - v_0\sin\theta\hat{y}\ .
\end{aligned} \tag{1}$$

We can add this to the center of mass velocity to find the total speed of each star relative to the cluster, after the interaction:

$$\begin{aligned}
|\mathbf{v}_1| &= \sqrt{(v_{\rm CM} + v_0\cos\theta)^2 + v_0^2\sin^2\theta} \\
&= \sqrt{v_{\rm CM}^2 + v_0^2 + 2v_0v_{\rm CM}\cos\theta} \\
|\mathbf{v}_2| &= \sqrt{(v_{\rm CM} - v_0\cos\theta)^2 + v_0^2\sin^2\theta} \\
&= \sqrt{v_{\rm CM}^2 + v_0^2 - 2v_0v_{\rm CM}\cos\theta}\ .
\end{aligned} \tag{2}$$

What does this mean? Suppose that $v_{\rm CM} > 0$ and $v_0 > 0$. From our definitions, that means that star 1 was moving faster than star 2, as seen in the cluster frame. Its original speed was $v_1 = v_{\rm CM} + v_0 = \sqrt{v_{\rm CM}^2 + v_0^2 + 2v_0v_{\rm CM}}$. After the interaction, unless $\cos\theta = 1$ (which would be equivalent to no interaction at all!), star 1 has a speed less than its initial speed. Star 2, on the other hand, has sped up. Therefore, energy has been transferred from the faster star to the slower star.

The idea that energy flows from the more energetic to the less energetic star is reminiscent of thermodynamics, and we'll hear a lot more about such parallels in future lectures. If one considers all angles of encounters, one finds that for equal-mass stars the "stable" distribution of velocities is a Maxwell-Boltzmann distribution, just as for molecules of a gas.

Now we get to return to binaries. Suppose that one has a "soft" binary, where both stars are of equal mass, which is the same as the mass of the single stars. By "soft" in this circumstance we mean that the speed of the two stars in their binary orbits is less than the speed of a single star in the cluster. If we go through conservation of energy and angular momentum, we conclude that an encounter between this binary and a single star could have any of the four allowed outcomes we discussed earlier. However, in reality the tendency is for the binary to become wider, i.e., softer, after the encounter. Let's see why.

Consider the limit of a very soft binary, and imagine a single star passing much closer to one star than to the other. A really soft binary orbits very slowly, so we'll ignore the orbit and just imagine that the single star has a quick close encounter with the closer star. When this happens, since the single star is moving faster than the binary star, the tendency is for the binary star to gain energy. But a larger positive energy means a wider binary orbit, given that $E = -Gm_1m_2/(2a)$. Therefore, typically, soft binaries get softer.

Another important point is this: I assert that for a very soft binary, the typical change of energy in an encounter with a single star does not depend on *how* soft the binary is. **Ask class:** why would that be true? From the treatment above, when $v_2 \ll v_1$, $v_{\rm CM} \approx v_1/2$ and $v_0 \approx v_1/2$, so the exact value of v_2 doesn't matter. Only v_1 and θ matter. Therefore, the change in energy is roughly independent of the original binding energy of the binary. **Ask class:** what does this imply about the eventual fate of soft binaries? Since energy keeps getting pumped into the binaries, eventually the total energy will be positive, meaning that the binary is disrupted! This is called evaporation. A very soft binary tends not to last very long.

What about the other limit, of hard binaries? Again, consider the interaction between a single star and a binary composed of two stars each having the same mass as a single star. Define "hard binary" as a binary having a binding energy greater than the kinetic energy of the single star. **Ask class:** what are the possible outcomes of such an interaction? Since the total energy of the system is negative, we can't have an "ionization" in which all three go their way separately. The final result must be a single and a binary, although which stars end up in the binary depends on details.

Those details, unfortunately, absolutely require a computer. In the soft binary case we could make the approximation that the binary doesn't orbit during an interaction with a single star. Here, the binary goes through at least one orbit during the interaction. In fact, numerical simulations show that sometimes there can be tens, hundreds, even thousands of orbits before the system finally resolves itself into a binary and a single (these are called "resonant" interactions). What are we supposed to do? Luckily, analysis of endless numbers of simulations have revealed useful rules of thumb. That is, there are tendencies, but not absolutes.

One such tendency is that after the interaction, the binary tends to be tighter than it was originally. For three equal-mass stars, the semimajor axis shrinks by an average of about 20%. Again, this is not an absolute; it is possible that the binary will expand after a particular interaction, even if the binary is hard. However, usually the binary gets more compact. Heuristically, you can think of this as follows: the typical ejection speed of the single (when the encounter is finally over) is of order the binary orbital speed. Since the binary is hard, the orbital speed is greater than the speed that the single star had initially. Therefore, the single star has gained kinetic energy. Conservation of energy then means that the binary has a more negative orbital energy, therefore it is tighter. Combined with the rules for soft binaries, we have "Heggie's laws" (named after Douglas Heggie, a pioneer of such studies): hard binaries get harder, and soft binaries get softer.

Another tendency is that if the three stars are of unequal mass, then most probably the final binary will consist of the two most massive of the three stars, if the interaction was a close interaction. Among other things, this means that heavy things in clusters (such as black holes) have a good shot of ending up in a binary even if they started their lives alone.

There is wonderful richness and complexity in three-body interactions, and things are being discovered even today. In the next couple of lectures, we'll go beyond three bodies to see what tendencies exist when a large number of objects interact gravitationally.