Three objects; “2+1” problem

Having conquered the two-body problem, we now set our sights on more objects. In principle, we can treat the gravitational interactions of any number of objects by simply adding together all the forces; for example, for n objects, the net force on object j is

$$\mathbf{F}_j = -\sum_{i\neq j}^{n} \frac{Gm_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i) \ . \tag{1}$$

Unfortunately, for $n > 2$ there is no general analytical solution. That means we need to do the integrations numerically if we have an arbitrary collection of three objects. This is a classic situation in astrophysics: the real problem we want to do is too complicated for analytical approaches. Do we throw our hands up in despair? No! Instead, we try to gain further insight by looking for simplified cases we *can* treat.

One such case is where one of the three objects has a tiny mass compared to the other two. We can then make the approximation that the two big guys are essentially not affected by the small mass. The motion of the two larger masses is then just what it would be in the two-body problem. The motion of the third mass can then be calculated in the field of force produced by the motion of the other two.

Suppose that the distance of the third mass from the other two is much greater than the separation between the two large masses. **Ask class:** to a reasonable approximation, what is the motion of the system? The two inner objects orbit each other as usual, and the third object orbits their center of mass as if it were a point mass, to lowest order. In fact, this is also true for arbitrary masses. We'll get to this case more in the next class.

But what if the third mass is close to the other two? In that case, even with the mass restrictions, if the two large bodies move in significantly eccentric orbits then the motion of the third mass is quite complicated in general. We therefore simplify further, and assume that the two large masses orbit each other in circles. Does this have applications in the real world? Yes! Close binary stars tend to circularize their orbits because of tidal effects. In our Solar System, most of the planets are on nearly circular orbits. It used to be thought (and may be true) that this was the result of dissipation in the protosolar disk, but since quite a few extrasolar planets have highly eccentric orbits this has to be rethought. In any case, nearly circular orbits are common enough that our approximation will indeed have applications.

Ask class: given our approximation, what is the motion of the two large masses? Since the motion is circular, they move at a constant angular velocity $\Omega = \sqrt{G(m_1 + m_2)/a^3}$, and maintain a constant separation. But what happens to a third object? If you just place it in the system with some arbitrary position and velocity, it will get batted all around the place by the time-varying forces in the system. Fun, but complicated. We can, however, get more insight by considering whether there are any special orbits that can allow a third mass to remain stationary with respect to the two large masses.

To do this, we'll shift to a coordinate system, centered on the center of mass of the two large objects, that rotates with the angular speed Ω. Because the orbits are circular, in this system the two large masses are fixed at a constant distance from each other. Let's set up axes so that the orbit is in the $x-y$ plane, and the masses are both on the x axis, with m_1 at $x=-a_1$ and m_2 at $x=+a_2$, with $a_1=m_2a/(m_1+m_2)$ and $a_2=m_1a/(m_1+m_2)$. What we'd like to do is find places in the $x-y$ plane where a third particle experiences no net acceleration. We have to realize, however, that in this rotating coordinate system there is an extra effective acceleration that is added. This is a centrifugal acceleration; if the particle is at location $\mathbf{r}$ then the centrifugal acceleration in this noninertial reference frame is $\Omega^2\mathbf{r}$. Generally, if mass m_1 is at location $\mathbf{r}_1$ and mass m_2 is at location $\mathbf{r}_2$, then the net acceleration in the rotating frame is

$$\mathbf{a}_{\rm net}=-\frac{Gm_1}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1)-\frac{Gm_2}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2)+\Omega^2\mathbf{r}\ . \tag{2}$$

We'd like to know points where $\mathbf{a}_{\rm net}=0$. To put it another way, in this rotating frame there is an *effective potential* that includes the effects of centrifugal acceleration. Let Φ represent the effective potential energy per unit mass, so that $\mathbf{a}_{\rm net}=-\nabla\Phi$. We then have

$$\Phi=-\frac{Gm_1}{|\mathbf{r}-\mathbf{r}_1|}-\frac{Gm_2}{|\mathbf{r}-\mathbf{r}_2|}-\frac{1}{2}\Omega^2r^2\ . \tag{3}$$

Effective potentials are common in physics, from condensed matter to general relativity.

We'll solve the equations in a moment, but first let's use our intuition to guess where such points might be. By symmetry, we notice that along the x axis there can be no net acceleration in the $+y$ or $-y$ directions. Therefore, if there are places along the x axis where there is zero net acceleration in the x direction, the acceleration is zero, period, and we have a point. Where are such places?

Let's start with x being large and negative. Clearly, for large $|\mathbf{r}|$, the centrifugal term (linear in r) dominates over the gravitational terms (inverse square in r). Therefore, when x is large and negative the net acceleration is in the $-x$ direction. **Ask class:** what is the net acceleration when the particle is to the left of m_1 but very close to it? If it's close enough then the inverse square term dominates, so the acceleration is in the $+x$ direction. That means that somewhere between $x\rightarrow-\infty$ and $x=-a_1$, the net acceleration must pass through zero. **Ask class:** moving along the x axis, what other points of zero acceleration can we identify in the same way? There must also be one between m_1 and m_2, and one to the right of m_2. By similar arguments, you can also show that there must be two zero acceleration points off the x axis. These five points are called the *Lagrange points*. If

$m_2 < m_1$, then L_1 is between m_1 and m_2; L_2 is to the right of m_2 in our diagram; L_3 is to the left of m_1; L_4 is the off-axis point in the direction of rotation; and L_5 is the off-axis point in the direction opposite to rotation. All other points in the system experience net acceleration.

Incidentally, this method of identifying points is also useful any time you want some insight into an algebraic equation $f(x) = 0$ of cubic or higher order (when direct solution is messy). If you can show that $f(x)$ is negative in one place and positive in another, then it must have a root in between if it is an algebraic equation with positive powers of x.

Let us now solve the equation directly. We have

$$0 = -\frac{Gm_1}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}-\mathbf{r}_1) - \frac{Gm_2}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}-\mathbf{r}_2) + \Omega^2\mathbf{r} \; . \tag{4}$$

We have a vector equation in two dimensions, so this encodes two separate equations. We could do this in Cartesian $x - y$ coordinates, but for reference we'll instead use a method that is often helpful in dealing with vector equations. We want to look at the components of this equation that are perpendicular to, then parallel to, a given vector, in this case $\mathbf{r}$. **Ask class:** how can we project out the component of this equation that is perpendicular to $\mathbf{r}$? We can take the cross product of the whole equation with $\mathbf{r}$, which gives us just the perpendicular component. Doing this gives

$$\frac{Gm_1}{|\mathbf{r}-\mathbf{r}_1|^3}(\mathbf{r}_1\times\mathbf{r}) + \frac{Gm_2}{|\mathbf{r}-\mathbf{r}_2|^3}(\mathbf{r}_2\times\mathbf{r}) = 0 \; . \tag{5}$$

Suppose that $\mathbf{r}$ makes an angle θ with the x axis. Then, since $\mathbf{r}_1 = -a_1\hat{x}$ and $\mathbf{r}_2 = a_2\hat{x}$, we have

$$\mathbf{r}_1\times\mathbf{r} = -a_1 r\sin\theta\hat{z} \; , \qquad \mathbf{r}_2\times\mathbf{r} = a_2 r\sin\theta\hat{z} \; . \tag{6}$$

Therefore,

$$\frac{Gm_1a_1}{|\mathbf{r}-\mathbf{r}_1|^3} r\sin\theta = \frac{Gm_2a_2}{|\mathbf{r}-\mathbf{r}_2|^3} r\sin\theta \; . \tag{7}$$

On the x axis ($\theta = 0, \pi$) or at the center of mass ($r = 0$) this equation is satisfied automatically. If neither condition holds (as it doesn't for L_4 or L_5) then we notice that $m_1a_1 = m_1m_2a/(m_1+m_2) = m_2a_2$, so we're left with

$$|\mathbf{r}-\mathbf{r}_1| = |\mathbf{r}-\mathbf{r}_2| \; . \tag{8}$$

Therefore, L_4 and L_5 must be on the midline between the two masses, and the x component of the distance to m_1 and m_2 is just $a/2$. If the third particle has the y-coordinate y, then $d \equiv |\mathbf{r}-\mathbf{r}_1| = |\mathbf{r}-\mathbf{r}_2| = (a^2/4+y^2)^{1/2}$.

Now let's return to our full equation for zero acceleration. **Ask class:** how do we project out the component that is parallel to $\mathbf{r}$? We take the dot product with $\mathbf{r}$. We can then divide by r^2 and rearrange to get

$$\frac{G(m_1+m_2)}{a^3} = \frac{Gm_1}{d^3}(1+a_1/r) + \frac{Gm_2}{d^3}(1-a_2/r) \ . \tag{9}$$

As before, the m_1a_1 and m_2a_2 terms on the right hand side cancel each other. We therefore have

$$\frac{G(m_1+m_2)}{a^3} = \frac{G(m_1+m_2)}{d^3} \ . \tag{10}$$

Therefore, $d = a$. This says something remarkable: *the L_4 and L_5 points make an equilateral triangle, of side a, with m_1 and m_2 in the plane of the orbit*! This is true regardless of the mass ratio m_2/m_1. If $m_1 \gg m_2$ (as for the Sun and Jupiter), the L_4 point is 60° ahead, and the L_5 point is 60° behind, the smaller object in its orbit.

There aren't such simple solutions for the other three Lagrange points. In fact, you end up with a fifth-order algebraic equation, which does not have a general closed-form solution. However, as always, we can examine limiting cases. Let's consider L_1 first. **Ask class:** what is the simplest limiting case they can think of? When $m_1 = m_2$, by symmetry the L_1 point must be exactly in the middle, which is the center of mass. Now what if $m_2 \ll m_1$? Then the transition from m_1 to m_2 being the primary source of gravity (and therefore the net acceleration going from $-x$ to $+x$ in our diagram) must be pretty close to m_2. We can solve for this by saying that L_1 is at location $x = a_2 - R$, where $R \ll a$, and getting R to first order. If we do this we find $R \approx (m_2/3m_1)^{1/3}a$. We also find that L_2 is the same distance to the right of m_2. Therefore,

$$\begin{aligned} x_{L_1} &\approx a_2 - (m_2/3m_1)^{1/3}a \\ x_{L_2} &\approx a_2 + (m_2/3m_1)^{1/3}a \ . \end{aligned} \tag{11}$$

The radius $R = (m_2/3m_1)^{1/3}a$ is called the *Roche radius* or *radius of the Hill sphere*, depending on whether one is working in stellar or planetary applications. This radius is of great importance in many branches of astrophysics. That's because, if a body is dominated by gravity instead of material strength, if its size is greater than R it will get torn apart. Thus, if a star in a binary expands beyond R, some of its mass will flow onto its companion; this mass transfer underlies many phenomena such as cataclysmic variables, symbiotic stars, and X-ray binaries. Another application is to asteroids. Mounting evidence suggests that asteroids aren't solid bodies but are instead gravitational aggregates, or rubble piles (think of a heap of gravel several kilometers across). When an asteroid gets too close to a large body, the Roche radius or Hill sphere becomes smaller than the asteroid, so that it is stretched by tidal forces. Derek Richardson and his students Zoë Leinhardt and Kevin Walsh are working on applications of this effect to many phenomena.

Our last investigation of the Lagrange points has to do with their stability. Formally, the Lagrange points are infinitesimal points. At those points, $\nabla\Phi = 0$ so there is zero acceleration. But in reality, there are all kinds of little perturbations that will occur: the gravity of other planets, radiation forces, or who knows what. It therefore makes a difference whether the equilibrium at a given Lagrange point is stable or unstable. We can determine this in the standard way from calculus: we take a second derivative of the effective potential and examine its characteristics. The result of this examination is that L_1, L_2, and L_3 are unstable, in that a slight perturbation away from those points will grow large. In contrast, L_4 and L_5 are stable; a slight perturbation will cause a particle to orbit those points. Thus, L_4 and L_5 are collection points. For the Sun-Jupiter system, they are called the Trojan points, and asteroids there are called the Trojan asteroids. There are probably a greater number of asteroids at the Trojan points than in the "main" belt between Mars and Jupiter! In the Earth-Moon system, the eccentricity is high enough that there is not a significant collection of debris at our L_4 and L_5.

Even the unstable Lagrange points have their applications. Since the net acceleration is zero at, e.g., L_2, a spacecraft flown there needs to do relatively little course correction. Think of a pencil balanced on its point: it's unstable, yes, but small tweaks keep it balanced. By comparison, a pencil at a 45° angle needs constant strong maintenance. Since the same goes for spacecraft, there are plans for many spacecraft (e.g., the Next Generation Space Telescope and LISA) to be put at the Earth-Sun L_2 point.