One-body gravitation

Now we're finally ready to take on a specific force. To quote the great poet Arnold Schwarzenegger, "What is important is gravity". Universal gravitation was suggested by Isaac Newton, and represented an important transition in the way people thought about nature. Prior to Newton, many people if asked would have said that the governing laws of the heavens were different than the laws of the Earth. Afterwards, the search for unity went ahead full steam.

Newton's law says that if two objects of masses m_1 and m_2 are at locations $\mathbf{r}_1$ and $\mathbf{r}_2$, then object 2 attracts object 1 with a force

$$\mathbf{F}_{21} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}} \tag{1}$$

where $\mathbf{r} \equiv \mathbf{r_1} - \mathbf{r_2}$, $r = |\mathbf{r}|$, and $\hat{\mathbf{r}} \equiv \mathbf{r}/r$. **Ask class:** in what ways does this satisfy the constraints on forces that we discussed last time? The force on 2 due to 1 is equal and opposite to that on 1 due to 2, and the force is directed along the line between the two objects. Note also that the force only depends on $\mathbf{r_1} - \mathbf{r_2}$, and not the two positions separately, as is intuitively reasonable. The constant $G = 6.67 \times 10^{-8}$ cm^3 g^{-1} s^{-2} has units that are notoriously difficult to remember. The easiest way is to remember a formula involving G (such as the force formula above!), and work it out from the known units of force, mass, and distance.

This one little simple force underlies all the marvelous richness of gravity. In the next several classes we'll see how intricate gravitational interactions can be, but following the principle of "do the simplest thing first" we'll look at a stripped-down system. In particular, let's imagine that we have two point masses, so we don't worry about their interior structure. Let's also assume that one mass (say, m_1) is much greater than the other mass. Then, from Newton's second law, the acceleration of mass 1 is tiny by comparison to the acceleration of mass 2. To a first approximation, then, we can assume that object 1 is stationary and object 2 moves around it. This is a good approximation for star-planet systems, although deviations from this are what have allowed detection of extrasolar planets.

We will eventually need to do detailed derivations to determine the nature of the orbits. First, though, let's see what we can understand qualitatively. I assert that this orbit will take place in a plane. **Ask class:** what is an easy way to see this? It comes from symmetry. Imagine that the orbit starts off in a plane, which we'll call the x-y plane for convenience. The symmetry about that plane means there is no reason for the orbit to move into +z or -z; therefore, it'll stay in the original plane of orbit. What about the angular momentum of the orbit of m_2? The total angular momentum of the system is conserved as always, but as we know, in general a given component need not be conserved. **Ask class:** what can we say in this case? Here, the torque on m_2 from m_1 is $\mathbf{N} = \mathbf{r} \times \mathbf{F}_{12}$. But $\mathbf{F}_{12} \propto \mathbf{r}$, so $\mathbf{N} = 0$. Therefore, the angular momentum of the orbit of m_2 itself is indeed conserved. Incidentally, note that angular momentum conservation means that the orbit stays in a plane; if the orbital plane tipped, so would the angular momentum vector.

By the way, you can see that angular momentum conservation doesn't require an inverse square law per se. All it requires is that the force be a central force (directed towards the center, depending only on the distance from that center). Keep that in mind when we explore other central forces.

Okay, time to get into more detail. We'll start by proving a couple of important theorems.

Theorem 1: a particle inside a spherical shell of uniform density experiences no net force. This is most easily demonstrated using symmetry. Suppose the shell has radius R and mass per unit area Σ. Consider a point that is somewhere in the interior. Consider a solid angle $\delta\Omega$ from that point to the edge in one direction. Let the distance from the point to the edge be r_1. Therefore, the area of the shell intercepted is $\delta A_1 = r_1^2 \delta\Omega$, and the mass is $\delta M_1 = \Sigma \delta A_1 = \Sigma r_1^2 \delta\Omega$. If you imagine that there is a tiny mass m at the point in question, the magnitude of the gravitational force on that mass is

$$\delta F_1 = Gm\delta M_1/r_1^2 = Gm\Sigma\delta\Omega\ , \tag{2}$$

where the r_1^2 have cancelled. Now consider the force from the shell in the solid angle $\delta\Omega$ in exactly the opposite direction from the first piece. It, too, will exert a gravitational force of magnitude $GM\Sigma\delta\Omega$, but the direction will be opposite to that of the first piece. Therefore, combined, the forces from the two pieces will cancel. This can be done for any pair of points, hence the net force is zero from the whole shell. If the Earth were hollow, you'd basically float around inside as if you were in empty space.

Theorem 2: A particle outside a sphere experiences the same force as if all the mass of the sphere were concentrated at a point at the sphere's center. This is an extremely important theorem. We'll do it with calculus, but you get extra respect for Newton when you realize he had to do it with geometry (since he himself invented calculus, other people weren't used to it yet!). To prove this, let's show that the force from an infinitesimally thin spherical shell reduces to that of a point at the center. Then, we can build up a sphere from spherical shells.

Suppose the shell has a mass per area Σ, and a radius R. Set up a coordinate system where the center of the sphere $\mathcal{O}$ is at the origin and the point $\mathcal{P}$ at which we'll calculate the force is on the x axis, a distance r from $\mathcal{O}$. **Ask class:** by symmetry, what is the only possible direction of a net force? Along the x axis, as we've defined it, since in the y and z directions the forces will cancel. Consider a small element of the shell, an angle θ from the x axis as measured from $\mathcal{O}$. Let this element have an angular extent $d\theta$ as measured from

$\mathcal{O}$. Suppose that the center of the element is the point $\mathcal{Q}$. We define the distance $\mathcal{PQ}$ to be s, and let ϕ be the angle $\mathcal{OPQ}$ (i.e., the angle from the x-axis as seen from the point $\mathcal{P}$).

As defined, the force on a particle of mass m at $\mathcal{P}$ from the entire ring at θ is $\delta F = Gm\delta M/s^2 = Gm2\pi\Sigma R^2 \sin\theta d\theta/s^2$. However, we are only interested in the force along the x-axis, by symmetry, so we need to multiply this by $\cos\phi$. The total force is this component integrated over the shell, i.e., integrated over θ. Thus,

$$F = Gm2\pi\Sigma R^2 \int_0^\pi \frac{\sin\theta\cos\phi d\theta}{s^2}\ . \tag{3}$$

If you write this out it looks nasty, but you can simplify things by substituting and using s as the primary variable. From the law of cosines,

$$r^2 + R^2 - 2rR\cos\theta = s^2\ , \tag{4}$$

so if we differentiate and realize that r and R are constant, we get

$$rR\sin\theta d\theta = sds\ . \tag{5}$$

The limits of the integral, which previously were $\theta = 0$ to $\theta = \pi$, now become $r - R$ to $r + R$. Another application of the law of cosines gives

$$\cos\phi = \frac{s^2 + r^2 - R^2}{2rs}\ . \tag{6}$$

The integral then becomes

$$F = Gm2\pi\Sigma R^2 \int_{r-R}^{r+R} \frac{s^2 + r^2 - R^2}{2Rr^2s^2} ds\ . \tag{7}$$

If the mass per area of the shell is Σ, then the total mass M of the shell is given by $M = 4\pi\Sigma R^2$, so let's substitute that in. With a little rearrangement of the integral, we then have

$$\begin{aligned} F &= \frac{GmM}{4Rr^2} \int_{r-R}^{r+R} \left(1 + \frac{r^2-R^2}{s^2}\right) ds \\ &= \frac{GmM}{r^2}\ . \end{aligned} \tag{8}$$

Hooray! That simplifies things a *lot*. **Ask class:** what can we conclude about the force outside a sphere of *nonuniform* density, if the density distribution is spherically symmetric (i.e., the density can be a function of radius but nothing else)? It's still the same as a point mass at the center, since again we can break down the force into uniform spherical shells.

Given those two theorems, let's idealize the Earth as a sphere of uniform density. **Ask class:** if you drilled a narrow hole deep into the Earth, what would happen to your weight as you went further down? The part outside where you are doesn't contribute, so ignore that; it's only the mass interior to you that matters. For uniform density, a distance r from the center the mass is $M = \rho V \sim r^3$. The force scales as $M/r^2 \sim r$, so as you go deeper, r decreases and so does the force. It must be that way: think of the limit, when you go to the center $r = 0$. The whole Earth is exterior to you, so the force must vanish.

To a high degree of accuracy, stars and planets are spheres, so the second theorem eases our life greatly. Smaller objects, such as asteroids, can be significantly aspherical, so it's tougher to treat the gravitational field near them.

We will now specialize to the orbit of a particle with very small mass m around a star or planet with mass $M \gg m$ that can be approximated as a sphere, so that its gravitational field is that of a point. This case is most easily treated in cylindrical coordinates. Mind you, it doesn't *have* to be treated in cylindrical coordinates. Just as $\mathbf{F} = m\mathbf{a}$ will work for any mechanics problem, Cartesian coordinates can be used for any problem whatsoever. It's just easier in some cases to try another system (and it can give us more insight, too). This is like the joke told by Abraham Lincoln: "If you call a tail a leg, how many legs does a dog have? Four. Calling a tail a leg doesn't mean it is one." In cylindrical coordinates, the force equation reads

$$m\ddot{\mathbf{r}} = -(GmM/r^2)\hat{\mathbf{r}} \tag{9}$$

where the double dot over the r means a second time derivative. Note that we can cancel m out of this problem entirely. That's a rather profound physical statement. A 1 kg mass will orbit the Earth in exactly the same way as a 2 kg mass, and independent of, e.g., its composition. In general relativity this is a reflection of a deep principle called the equivalence principle. In any case, it is useful to break this equation into radial and transverse components. In cylindrical coordinates, the radial component of $\ddot{\mathbf{r}}$ is $\ddot{r} - r\dot{\theta}^2$, and the transverse component is $2\dot{r}\dot{\theta} + r\ddot{\theta}$. If you're not familiar with derivatives in cylindrical coordinates, I strongly recommend looking it up in your favorite book. This, by the way, is an example where looking at limits is a great way to get insight. For example, if there is only radial motion ($\dot{\theta} = 0,\ \ddot{\theta} = 0$), then only the radial component is nonzero, and its value is $\ddot{r}$ as it should be.

The radial and transverse components of the force equation are then

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= -GM/r^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} &= 0\ . \end{aligned} \tag{10}$$

Taking the second equation and multiplying by r we get a perfect differential, so it reduces to $d(r^2\dot{\theta})/dt = 0$. Therefore, $j \equiv r^2\dot{\theta}$ is a constant. **Ask class:** what is this constant? Since $r\dot{\theta}$ is the component of the velocity perpendicular to r, $r^2\dot{\theta}$ is the magnitude of $\mathbf{r} \times \mathbf{v}$, or the angular momentum per mass. Therefore, this is just the conservation of angular momentum. Incidentally, in planetary astronomy, what we've called j is usually called h. Sometimes it is called Λ. We're using j for consistency with our book. Note also that $r^2\dot{\theta}$ is the rate at which area is swept out by the orbit. We have therefore derived *Kepler's second law: equal* *areas are swept out in equal times by an orbit.*

To evaluate the radial component of our equation, it is convenient to use the new variable

$$u \equiv 1/r \ . \tag{11}$$

We find

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta} = -j\frac{du}{d\theta} \ . \tag{12}$$

The last step applies because $\dot{\theta} = ju^2$. Our second derivative is

$$\ddot{r} = -j\frac{d}{dt}\frac{du}{d\theta} = -j\dot{\theta}\frac{d^2u}{d\theta^2} = -j^2u^2\frac{d^2u}{d\theta^2} \ . \tag{13}$$

Substituting into our radial force equation, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{j^2} \ , \tag{14}$$

for which the general solution is

$$u = A\cos(\theta - \theta_0) + GM/j^2 \ . \tag{15}$$

Ask class: what is the physical significance of θ_0? It simply determines the orientation of the orbit relative to some arbitrary angle of reference. This means that we can select $\theta_0 = 0$ without loss of generality, so the equation for r becomes

$$r = \frac{1}{A\cos\theta + GM/j^2} \ . \tag{16}$$

We can further simplify this expression by defining

$$e \equiv Aj^2/(GM) \qquad r_0 \equiv j^2/[GM(1+e)] \ . \tag{17}$$

We then have

$$r = r_0\frac{1+e}{1+e\cos\theta} \ . \tag{18}$$

This is the polar equation of a conic section (circle, ellipse, parabola, hyperbola). This equation is written in such a way that the center of the coordinate system is at one focus, not the center, hence in a system with one dominant mass (e.g., the Sun in the Solar System) that mass is at a focus. Here r_0 is the distance of closest approach (called a pericenter in general; perihelion for the Sun, perigee for the Earth, and so on) and e is the eccentricity. For a circle, $e = 0$; for an ellipse, $0 < e < 1$; for a parabola, $e = 1$, and for a hyperbola, $e > 1$. Planets have $e < 1$, so we have derived *Kepler's first law: planets travel in ellipses.* Most planets have low-eccentricity orbits. Mercury (e=0.206) and Pluto (0.249) have the highest e. This explains why it took so long to discover that the planets weren't moving on epicycles on top of circles: for low e, an ellipse is well described by epicycles, to the accuracy of observation available to the ancients.