Radiation Part 2: Random Walks and Doppler Shifts

Here we talk in some detail about two subjects that are quite important in the understanding of radiation. The first involves scattering, in the form of what are called random walks. The second gives the full formula for how photons appear to have different frequencies depending on how the source is moving relative to the observer.

1. Random Walks

Suppose we have a photon inside a star, or inside a molecular cloud. If it is deep enough inside, then it will probably scatter, get absorbed, and have another photon re-emitted many times. Thus it will bounce around many times before it finally leaves. This is sometimes analogized to the situation of someone who is completely drunk and thus takes one step in a random direction and then another in a different random direction. You might think that they wouldn't be able to make any progress, but in fact they do tend to move away from where they started. In a similar fashion, a photon bouncing around randomly won't have a particular direction that it is going, but it will tend to move away from its origin point.

1.1. Mean free path and optical depth

To ramp up to that discussion, we need to define a few concepts. The first is the *mean free path*, for which we will often use ℓ. To understand the mean free path, think about a photon traveling in a region of space that has a number of identical asteroids distributed randomly. The randomness of the distribution means that for some photon paths, the photon will hit an asteroid quickly; for others, the photon will travel quite a distance before it hits an asteroid. But averaged over all directions and initial locations of the photon, there will be some mean distance to travel before the photon hits an asteroid. This is the mean free path ℓ.

What does the mean free path depend on? Intuitively, the bigger the asteroids are (all else being equal), the shorter the mean free path; there's more to hit. Similarly, it seems that if you fix the size of the asteroids but increase their number density (which is their average number per volume), the mean free path should decrease as well. It's the area of the asteroids that matters; think about looking out into a sky of asteroids, where the distance you can see in a given direction depends on how much of your two-dimensional sky out to some distance is covered with asteroids. Thus one of the quantities you care about is the area of the asteroids; and not just the area, but the area in projection (e.g., πR^2 for a sphere, rather than the full $4\pi R^2$). This area is called the *cross section*, and it is usually designated by σ. In SI units, σ has the usual units of area: m^2.

The number density is a number per volume, which means that its units in SI are m^{-3}. n is usually chosen to represent a number density.

We want to combine these to get a length. The only way to combine σ and n to get a length is $1/(n\sigma)$, so we can hypothesize that

$$\ell = \frac{1}{n\sigma} \ . \qquad (1)$$

This has the right units by construction. Does it satisfy our intuition about limits? From this formula, larger number density n does lead to lower mean free path, as expected, and larger cross section also leads to the expected lower mean free path. Believe it or not, that's all we need to do; this is the right expression!

Of course, in our main applications we're not thinking about photons interacting with asteroids. Instead, photons can interact with electrons (in or out of atoms), and sometimes with molecules. But the same principle applies. For a given type of interaction (e.g., scattering off an electron, or being absorbed by an atom), it is possible to calculate the cross section of that interaction for a photon of a given frequency. Mind you, the "of a given frequency" bit is important: photons of some frequencies interact very strongly with atoms because that frequency is exactly what is needed to bump an electron from one energy level to a higher energy level, whereas photons with frequencies just a bit lower or higher might barely interact at all.

Also, as always, we need to think carefully about whether our assumptions work for a particular case. For example, we assumed in our development above that the asteroids (or atoms or molecules) are distributed randomly in space. That's a fine approximation for the gases that we normally encounter in astronomy. But if the substance is a crystal, there can be some directions along which a photon can go for long distances without interacting, and some directions along which the photon is constantly bumping into things. Please always ask yourself "do the standard assumptions apply?" when you investigate a problem in astrophysics!

Now, suppose that the mean free path of your photon is ℓ and the distance through some region is D. If the mean free path is constant over the whole region, then if you somehow forced the photon to move in a straight line for the whole distance D but kept track of the interactions it had along the way, you would expect it to have D/ℓ interactions. This ratio is something to which we give the name *optical depth* τ:

$$\tau = D/\ell \ . \qquad (2)$$

The name "optical depth" makes it sound as if we are thinking only of optical light, but in fact this is the term used generally; we could apply it to X-rays, or radio waves, or even things that aren't photons (such as neutrinos).

Note that τ could be less than 1 (even much less than one) or greater than 1 (even much greater than one). For example, in most directions ordinary light travels across the Solar System with hardly any interactions, so in that case $\tau \ll 1$. In contrast, the mean free path to photons in the interior of the Sun might be 10^{-5} meters or less. Given that the Sun's radius is about 7×10^{8} meters, clearly $\tau \gg 1$ for that situation!

Note for those comfortable with calculus: in the above we assumed that the mean free path is constant over the whole region. That won't be true in general; for example, the Sun's number density certainly changes with distance from its center. If we think about the straight-line path through the region as being parameterized by x, and the mean free path at some point being given by $\ell(x)$, then we just have

$$\tau = \int_0^D \frac{dx}{\ell(x)} , \tag{3}$$

which is to say that we add up the optical depth little by little over the whole path.

In the limit $\tau \ll 1$, the region is called *optically thin*, and in the limit $\tau \gg 1$, the region is called *optically thick*. The probability that a photon can get completely through a region of optical depth τ without *any* interactions is $e^{-\tau}$. As always, it is useful to think about such a formula in its limits:

- If $\tau \ll 1$, then because $e^x \approx 1 + x$ when the absolute value of x is much less than 1 (try it!), then $e^{-\tau} \approx 1 - \tau$. Because the probability of getting through with no interactions is $1 - \tau$, that means that the probability of having at least one interaction is $1 - (1 - \tau) = \tau$.
- If $\tau \gg 1$, then $e^{-\tau}$ is much, much less than 1. For example, if $\tau = 10$, then $e^{-\tau} \approx 4.5 \times 10^{-5}$. It is *very* difficult to sail through a high optical depth region. That's why a wall protects you from sunlight :).

1.2. Random walks

Of course, if a photon is bouncing this way and that, then in general its path will *not* be a straight line. In this subsection, we will explore the average behavior of photons as they interact repeatedly on the way out of a region.

This moving around is usually approximated as (1) photon moves, (2) photon hits something, (3) after hitting something, photon moves in a completely random new direction, (4) process repeats. It is therefore commonly called a random walk.

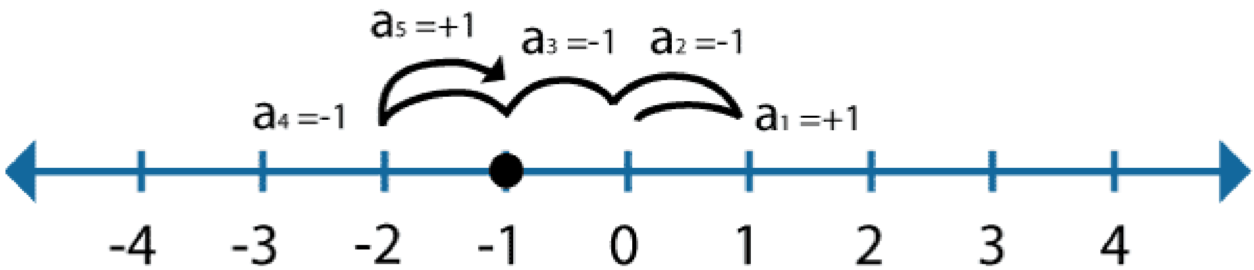

Fig. 1.— One-dimensional random walk. We imagine that the photon starts at the origin ("0") and takes a step left or right with equal probability, always with a distance of 1. Original figure from https://www.mit.edu/∼kardar/teaching/projects/chemotaxis(AndreaSchmidt)/random.htm, which has a lot of useful material related to random walks.

To gain insight, we will consider a very simplified case. We imagine that a photon is constrained to move left and right along a line. In a given step, a photon can move 1 unit of distance, and the direction is left or right with equal probability. See Figure 1 for a depiction.

We'd like to know how far the photon gets from its initial location; for example, how long will it take for a photon to escape from a gas cloud? That has relevance in a lot of astrophysical contexts. You might want to know how long it takes a cloud to radiate away its energy (i.e., to cool), which is important for our understanding of how stars form (because radiation from a cloud, which lowers the cloud's total energy, allows the cloud to condense and form stars). But most of the photons are deep in the cloud, at high optical depth from the surface, so understanding how they escape is key if we want to know how long it takes for the cloud to cool.

As a first step, we can try to figure out where we expect the photon to be after some number of steps. So let's do that from the beginning:

1. We start with the photon at location 0.

2. We have an equal probability of taking the next step to the left (to location -1) or to the right (to location 1). Because the probabilities are both equal to 1/2, the average location is $\frac{1}{2}(-1) + \frac{1}{2}(1) = 0$. After one step, the average location is still 0.

3. If the initial move was to -1, then the next move is again random: 1/2 probability to -2, and 1/2 probability to 0. This channel started with 1/2 probability, so after two moves there is a $(1/2) \times (1/2) = 1/4$ probability of going to -2, and a $(1/2) \times (1/2) = 1/4$ probability of going to 0. Likewise, if the initial move was to 1, then after two moves there is a $(1/2) \times (1/2) = 1/4$ probability of going to 0 and a $(1/2) \times (1/2) = 1/4$ probability of going to 2. Thus after two steps, the average location is $(1/4)(-2) + (1/4)(0) + (1/4)(0) + (1/4)(2) = 0$. After two steps, the average location is still 0.

4. In fact, you can convince yourself that after *any* number of steps, the average location is 0.

   Challenge: prove this!

So this doesn't sound very promising! The average location will always be the original location.

But what we'd really like to know is the average *distance* from the origin. Then whether the photon is at the negative part of the line or the positive part doesn't matter. For example, the average distance after one step is $\frac{1}{2}(1) + \frac{1}{2}(1) = 1$, not 0.

To figure out the average distance it turns out to be easiest to figure out the average of the *square* of the distance, and then take the square root of that average square. This is called the "root mean square" distance, and the reason it's easier is that squaring things prevents them from being negative :).

So let's start. After one step, the mean square distance is $\frac{1}{2}(-1)^2 + \frac{1}{2}(1)^2 = 1$. After two steps, the mean square distance is $\frac{1}{4}(-2)^2 + \frac{1}{4}(0)^2 + \frac{1}{4}(0)^2 + \frac{1}{4}(2)^2 = 1 + 1 = 2$.

So that's interesting. After one step the mean square distance is 1, and after two steps the mean square distance is 2. Let's make a jump; suppose that after N steps the mean square distance is N. If we can prove that *given this* after $N+1$ steps the mean square distance is $N+1$, then we have satisfied the conditions for mathematical induction and thus proved our hypothesis.

If we assume that after N steps the mean square distance is N, then the root mean square distance is $\sqrt{N}$. There is an equal probability of going to the left or to the right, so after one more step there is a 1/2 probability of moving to $\sqrt{N} - 1$, and a 1/2 probability of moving to $\sqrt{N} + 1$. Thus after one more step the mean square distance is

$$\frac{1}{2}\left(\sqrt{N} - 1\right)^2 + \frac{1}{2}\left(\sqrt{N} + 1\right)^2 = \frac{1}{2}\left(N - 2\sqrt{N} + 1\right) + \frac{1}{2}\left(N + 2\sqrt{N} + 1\right) = N + 1\ . \qquad (4)$$

Thus our expression works for any N. The average square distance after N steps is N, and thus the root mean square average distance after N steps is $\sqrt{N}$.

Let's be clear about what this means. If you take N steps of unit length, the *average* distance from where you started is $\sqrt{N}$. But you don't know where that will be. In our one-dimensional case, it might be that you end up left of the origin, or right of the origin. Also, this is just the *average*. Any given photon might end up closer or farther than $\sqrt{N}$ away from the origin. Thus this is a statistical statement.

But it seems that we have proven just a very limited result. In our derivation, we have made two highly restrictive assumptions: (1) the motion is one-dimensional, and (2) each step has length 1. Real photon motion is three-dimensional, and the distance traveled by a photon between interactions isn't the same all the time.

However, remarkably, you can show that this result is far more general than it appears. Even in three-dimensional motion, and even when the distance traveled between interactions varies (but with a mean free path ℓ), you can show that after many interactions $N \gg 1$, the root mean square distance from the origin is proportional to $\ell\sqrt{N}$ (the proportionality constant is close to 1 and depends on details).

What does this imply? If we go back to our definition of optical depth, we recall that it is the number of scatterings we *would* have had if the photon had somehow been forced to move in a straight line. Suppose that we would like to know how many interactions are typically required to go some distance $R \gg \ell$. Then

$$\begin{aligned} \ell\sqrt{N} &\approx R \\ N &\approx (R/\ell)^2 \\ N &\approx \tau^2\ . \end{aligned} \tag{5}$$

Thus if $R/\ell = \tau \gg 1$, it takes of order τ^2 interactions to escape. The total distance is then roughly $\tau^2\ell = \tau(\tau\ell) = \tau R$, which means that it can take a long time to escape from a highly optically thick medium.

2. Doppler Shifts

And now, a much shorter section about Doppler shifts, which we include here because we want to give a bit more detail than is available in our book.

As our book indicates, if a source of light emitting at some precise wavelength is moving toward us, then we see a higher frequency (and thus a shorter wavelength) than would be seen in the rest frame of the source. This is called a blueshift. In contrast, if the source of light is moving away from us, then we see a lower frequency (and thus a longer wavelength) than would be seen in the rest frame of the source. This is called a redshift, and the overall effect is the Doppler effect, or a Doppler shift. You're probably familiar with it from experience in listening to sirens: when the siren is coming toward you it has a higher pitch (and thus a higher frequency) than when it is moving away from you.

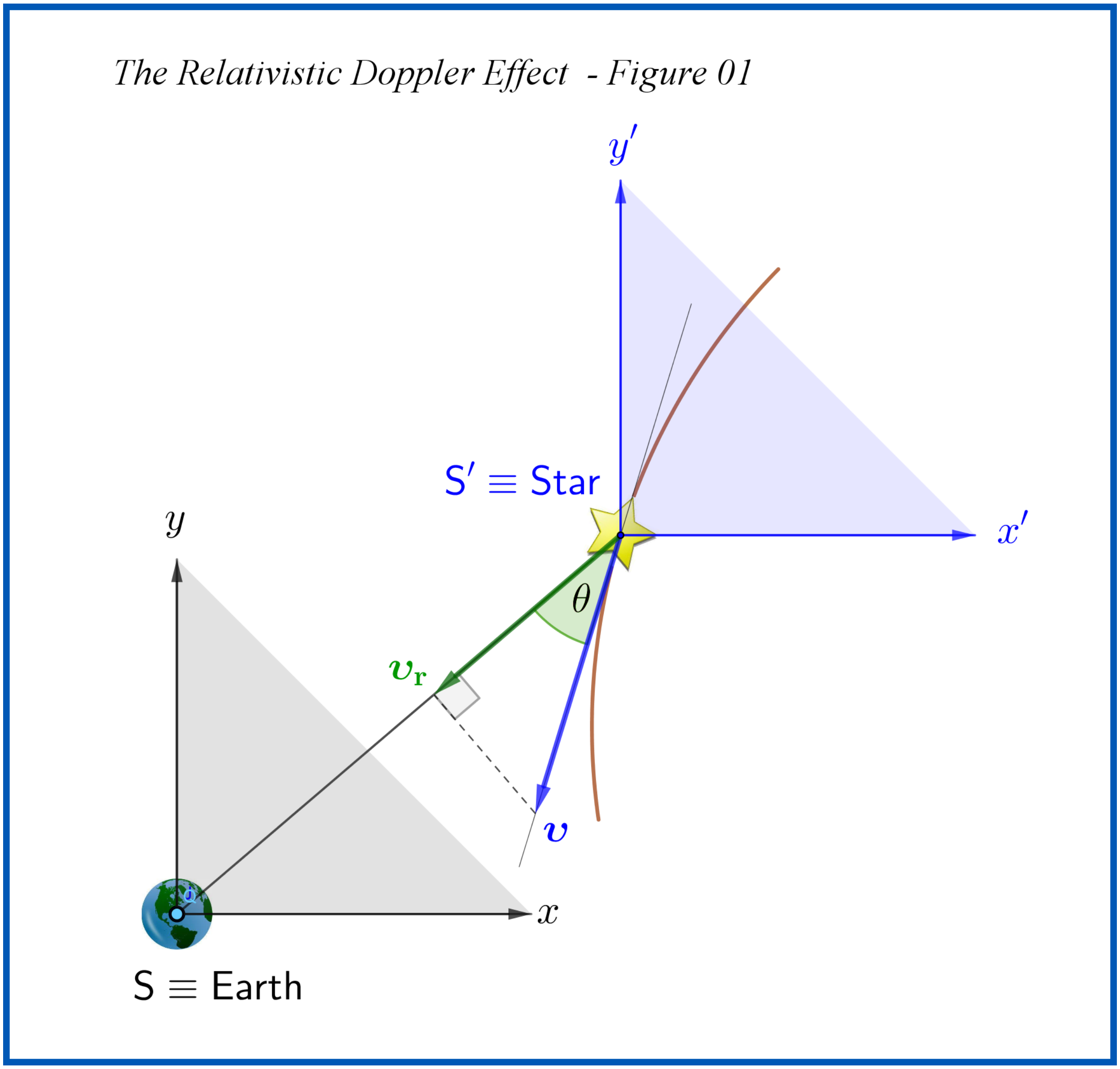

Fig. 2.— Geometry for a Doppler shift. The source is moving at a speed v, in a direction that is θ radians away from head-on to us. Original figure from https://i.stack.imgur.com/SRBgY.png.

The basic geometry is shown in Figure 2. The source (a star in the figure) is moving at a speed v, in a direction that is θ radians away from being directly at us. Suppose that in the rest frame of the source the emitted light has a frequency $f_{\rm emitted}$, and that we receive the light at a frequency $f_{\rm received}$. Then the relation between the two is

$$f_{\rm received}/f_{\rm emitted} = \gamma[1 + (v/c)\cos\theta] \ , \tag{6}$$

where c is the speed of light and γ is the *Lorentz factor*

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} . \tag{7}$$

For $v \ll c$, γ is close to 1. For example, the Sun orbits the center of our Galaxy at about 200 km s^{-1}, which might seem pretty fast, but because the speed of light is about $c \approx 300,000$ km s^{-1}, $\gamma \approx 1.00000022$. However, when we get to cosmology, the apparent recession speeds can approach the speed of light, at which point this makes a difference.

To reinforce the practice of working with equations to understand them, let's consider the case of motion straight toward us (thus $\theta = 0$) and straight away from us (thus $\theta = \pi$). For the formula to be correct, it must be that motion toward us gives $f_{\rm received}/f_{\rm emitted} > 1$, and that motion away from us gives $f_{\rm received}/f_{\rm emitted} < 1$. Let's see if that's true.

If $\theta = 0$ then $\cos\theta = 1$, and thus the ratio is $\gamma(1 + v/c)$. Looking at the expression for γ, we remember that in general $a^2 - b^2 = (a - b)(a + b)$, so $1 - v^2/c^2 = (1 - v/c)(1 + v/c)$. Thus

$$\begin{aligned} \gamma(1 + v/c) &= \frac{(1+v/c)}{\sqrt{1-v^2/c^2}} \\ &= \frac{(1+v/c)}{\sqrt{(1-v/c)(1+v/c)}} \\ &= \frac{\sqrt{(1+v/c)(1+v/c)}}{\sqrt{(1-v/c)(1+v/c)}} \\ &= \sqrt{(1 + v/c)/(1 - v/c)} . \end{aligned} \tag{8}$$

The speed v is never negative, so the ratio is greater than 1 unless the source isn't moving at all ($v = 0$).

Similarly, for $\theta = \pi$, $\cos\theta = -1$, so the ratio is $\gamma(1 - v/c)$. We thus get

$$\begin{aligned} \gamma(1 - v/c) &= \frac{\sqrt{(1-v/c)(1-v/c)}}{\sqrt{(1-v/c)(1+v/c)}} \\ &= \sqrt{(1 - v/c)/(1 + v/c)} , \end{aligned} \tag{9}$$

which is always less than or equal to 1. Therefore the expression conforms to our understanding of these limiting cases.

By the way, you can note that for motion perpendicular to our line of sight (that is, $\theta = \pi/2$), the ratio is $f_{\rm received}/f_{\rm emitted} = \gamma$. Because $\gamma \neq 1$ unless $v = 0$, this means that there is a *transverse Doppler shift* and that many textbooks are not quite accurate when they say that motion perpendicular to our line of sight does not change the wavelength. The wavelength does change, but typically by such tiny amounts that textbook authors can be forgiven for making the simplification :).

And, as always, feel free to talk with the tutors, the TAs, or me about the topics in this supplement!

Practice problems

1. In a gas cloud, the number density of molecules is $n = 10^{10}$ m^{-3} and the effective cross section of the molecules for a particular photon is $\sigma = 3 \times 10^{-20}$ m^2. Calculate the mean free path ℓ.

Answer: $\ell = 1/(n\sigma) \approx 3 \times 10^9$ m.

2. The gas cloud in the previous problem is a sphere with a radius $R = 1$ pc. Calculate the optical depth from the center of its cloud to the edge for the same photon as before.

Answer: one parsec is about $D = 1$ pc $\approx 3 \times 10^{16}$ m. The optical depth is $\tau = D/\ell \approx 10^7$. You can also get this answer from $\tau = n\sigma D$.

3. Do the same calculations as in the first two problems for roughly the average number density of the interstellar medium, $n \approx 10^6$ m^{-3}, and an effective cross section of about $\sigma = 10^{-20}$ m^2.

4. Do the same calculations as in the first two problems for roughly the average number density of the universe, $n \approx 0.3$ m^{-3}, and an effective cross section of about $\sigma = 10^{-20}$ m^2. For photons with that cross section do we expect to be able to see all the way across the universe for a typical line of sight?

5. Prove that even when the distance traveled between interactions varies (but with a mean free path ℓ, and even in three-dimensional motion, after many interactions $N \gg 1$ the root mean square distance from the origin is proportional to $\ell\sqrt{N}$. Probably the easiest way to do this involves vectors. The total vector displacement after N steps is $\vec{r}_{\rm tot} = \vec{r}_1 + \vec{r}_2 + \cdots + \vec{r}_N$. The squared distance is then $\vec{r}_{\rm tot} \cdot \vec{r}_{\rm tot}$, which involves terms such as $\vec{r}_1 \cdot \vec{r}_1$ and $\vec{r}_2 \cdot \vec{r}_2$, as well as cross terms such as $\vec{r}_1 \cdot \vec{r}_2$. If the vectors are all randomly oriented with respect to each other, what is the average value of $\vec{r}_1 \cdot \vec{r}_2$ and similar terms?

6. Suppose that the mean free path for a photon in the Sun is 10^{-5} meters. Calculate how long it would take the typical photon to escape from the center of the Sun. Please assume that scatterings are instantaneous and that between scatterings the photons travel at the speed of light.

7. The other day I was stopped for running a red light. Figuring that I would overawe the police officer with my physics knowledge, I told her that the Doppler shift as I approached the light head-on made it look green to me. Unfortunately, she was a former astronomy major at Maryland, and so she changed my charge to speeding. At a cost of \$100 per ten miles per hour over the posted speed limit of 35 mph, calculate to within 1% the cost of my ticket. For this, you may assume that a green traffic light (which I claim I saw) has a wavelength of 522 nanometers, and a red traffic light has a wavelength of 657 nanometers.