Light and the Properties of Electromagnetic Radiation

In this supplement, after reviewing what the book gives you about light, we add some details. In particular, we take a close look at blackbody radiation. One of the main points we want to get across is how to look at a complicated equation and derive physical insight using limits. Therefore, we will go into significant detail as we look at limits of the blackbody formula.

1. The Properties of Light

These supplements are meant to, well, supplement the textbook rather than replacing it. Therefore I will assume that you have read the textbook and that among other things you understand that:

1. Light and matter can interact in various ways, including emission (matter produces light), absorption (matter absorbs light), transmission (light passes through matter), and reflection (light bounces off of matter).
2. Light can be considered as a particle or as a wave; which point of view is more convenient depends on the circumstances.
3. In the particle framework, the particles of light are called *photons*. Photons have energy, linear momentum, angular momentum, frequency, wavelength, and polarization.
4. In a vacuum, all wavelengths of light travel at the same speed: the speed of light, $c = 2.99792458 \times 10^8$ m s^{-1} (this is actually exact, because a meter is defined as the distance light travels in a vacuum in $1/(2.99792458 \times 10^8)$ seconds!).
5. The wavelength λ, the frequency f, and the energy E of a photon are related by $f = c/\lambda$ and $E = hf$, where $h = 6.62607015 \times 10^{-34}$ m^2 kg s^{-1} is Planck's constant.
6. Atoms can interact with matter in various precise ways. For example, when an electron in an atom drops from a higher energy state into a lower energy state, this produces a photon with a very well-defined energy. Different atoms have different well-defined energies associated with them, and this allows astronomers to identify atoms and their properties, including their temperature and how they are moving with respect to us.

In addition, we note that light:

1. Has linear momentum. A photon of energy E has momentum $p = E/c$. The linear momentum of light means that photons can bounce off of things and deliver force!

2. Has angular momentum. The magnitude of the angular momentum of a photon, regardless of its energy, is $L = h/(2\pi)$; note that $h/(2\pi)$ is usually given the special symbol $\hbar$.

3. Has a universal speed (c) in vacuum, but *not* in matter. For example, light travels more slowly through glass or air than in vacuum (perhaps not surprising; light is an electromagnetic wave, and if the wave has to wiggle matter it can't travel as fast as when it's moving through vacuum), and in a medium (such as glass or air), it is usual that different frequencies travel at different speeds. This is what causes refraction. In astronomy, this effect causes lower-frequency radio waves to travel more slowly through the interstellar medium than higher-frequency radio waves. That effect, called *dispersion*, spreads out radio signals by wavelength, and can be used to make crude estimates of distances to sources.

2. Why light?

Most of our information about the universe comes from light, which is made up of individual photons. There are other messengers from deep space: neutrinos, gravitational waves, and high-energy charged particles (i.e., cosmic rays). Let's compare the properties of photons with those other messengers.

- Photons interact with things, but not too strongly. Neutrinos and gravitational waves sail through the universe with almost no interactions. That means that matter in the way essentially doesn't block them, and that in principle their directions and energies can tell us a lot about their sources. However, their very weak interactions also mean that for the most part they go through detectors with minimal interactions. That means that only a very small fraction of the energy in neutrinos and gravitational waves can be detected, and thus only exceptionally energetic events can be detected via these channels. Massive, electrically charged particles have the opposite problem. Electrons, protons, and nuclei can be accelerated to high energies, but they are curved by the magnetic field of our Milky Way galaxy (and the magnetic field between galaxies, if the source originates farther away), and slam into air molecules (or go all the way through detectors), so some information is lost. Again, it is typically only highly energetic sources that can be seen in highly energetic charged particles.

- All kinds of objects can emit photons. Heat is all that is needed, but many other processes produce photons as well (this is fundamentally because the electromagnetic interaction is pervasive and relatively strong). In contrast, significant production of gravitational waves requires fast motion of large masses, and production of high energy particles needs large electrical potentials or other acceleration mechanisms. Neutrinos are actually produced pretty commonly (hydrogen fusing into helium generates them), but not enough to compensate for their extremely weak interactions.

- Detectors can measure with precision many aspects of photons. These include energy, direction, time of arrival, and polarization. In principle these quantities can also be measured for the other messengers, but in practice such measurements are at much worse precision than is usually available for photons.

3. Blackbody Radiation

Suppose we have matter that is completely in equilibrium with itself and with the radiation around it. That means that all processes, and their inverses, are in balance, at least statistically. What we mean by that is that in the matter as a whole, every emission is balanced by a corresponding absorption somewhere else; there is no net emission or absorption from the overall matter.

We should understand that this is an *approximation*. For example, almost all photons from the photosphere of the Sun come to us without interacting further; for those photons, their emission is *not* balanced by a corresponding absorption.

But there *are* circumstances in which the assumption of equilibrium is very close to true. For example, in the interior of the Sun, photons that are emitted can't go very far before they are absorbed, and therefore it is a good approximation to say that in the Sun's interior the matter and the radiation are in equilibrium, which among other things means that locally the radiation and the matter have the same temperature T. This T changes with location in the Sun (T is much larger in the center of the Sun than it is halfway to the photosphere), but locally the matter and radiation are very close to equilibrium.

When the radiation is in equilibrium in this way, then the spectrum of the radiation is uniquely determined, and is *blackbody radiation*. Because this spectrum occurs in a variety of astrophysical contexts (one of the most remarkable being the cosmic microwave background!) it is therefore useful to study the blackbody spectrum in some detail.

The amount of energy emitted per time per area per frequency for a blackbody at a temperature T, at a frequency ν, is given by the function

$$B(\nu, T) = \frac{8\pi h\nu^3}{c^2} \frac{1}{e^{h\nu/k_B T} - 1} , \tag{1}$$

where $k_B = 1.38064852 \times 10^{-23}$ J K^{-1} is Boltzmann's constant. Thus when you multiply k_B by a temperature, you get an energy; because $h\nu$ is also an energy, this means that $h\nu/k_B T$ is dimensionless. It has to be, because the argument of an exponential or a trigonometric function such as a sine, cosine, or tangent has to be dimensionless (do you understand why?).

When we see an equation such as this, the full form may not give us much insight. It is therefore very helpful to look at limits.

What limits might we look at? Remember, we are thinking about the emission at some frequency ν for a temperature T. We could ask about what happens when $h\nu \ll k_BT$ (i.e., at low frequency for that temperature), and what happens when $h\nu \gg k_BT$ (i.e., at high frequency for that temperature). We'll do the low-frequency limit first.

We will focus our attention on the denominator, and in particular on $e^{h\nu/k_BT}$. So that we don't clutter things up too much, let's define $x \equiv h\nu/k_BT$. Therefore, $h\nu \ll k_BT$ means that $x \ll 1$. So what is e^x when $x \ll 1$? Something you learn in calculus is that when $x \ll 1$, $e^x \approx 1+x$. But here is a case where you can try it yourself. Using a calculator, what is $e^{0.1}$? How about $e^{0.01}$ or $e^{0.001}$? You should find that the smaller x is, the better e^x is approximated by $1+x$.

Thus if

$$e^x \approx 1+x \; , \tag{2}$$

then

$$e^{h\nu/k_BT} - 1 \approx 1 + h\nu/k_BT - 1 = h\nu/k_BT \; . \tag{3}$$

Substituting this back in to our original equation, we find that when $h\nu \ll k_BT$,

$$B(\nu,T) \approx \frac{8\pi h\nu^3}{c^2}/(h\nu/k_BT) = 8\pi\nu^2 k_BT/c^2 \; . \tag{4}$$

What do we make of this limit? One thing to note is that in this limit, the spectrum scales with frequency as ν^2. This limit, in fact, was derived well before the full form was known, and it poses a problem: if the spectrum were just proportional to ν^2 for all frequencies ν, then as $\nu \to \infty$ the emission would become infinite! That obviously doesn't make sense. Because ultraviolet light has a higher frequency than visible light, and because the divergence of the spectrum happens at high frequencies, this was called the *ultraviolet catastrophe* when it was the only spectrum derived for blackbody light. Incidentally, "The Ultraviolet Catastrophe" would be an awesome name for a band...

We'll get to the high frequency limit in a bit, but there is one other point to make that is not immediately evident. You can see that in the low-frequency limit, the final approximate expression does not involve Planck's constant h. This is meaningful because it turns out that the presence of h in a formula indicates that quantum mechanics is important. Thus what this formula tells us is that for low frequencies compared with the temperature, quantum mechanics is *not* important; indeed, the expression was derived before quantum mechanics existed.

Now let's turn our attention to the high-frequency limit, in which $x \equiv h\nu/k_BT \gg 1$. If $x \gg 1$, then e^x is much larger than x; for example, if $x = 10$, then $e^x \approx 2.2 \times 10^4$. Thus in this limit, $e^x - 1 \approx e^x$, and

$$B(\nu,T) \approx \frac{8\pi h\nu^3}{c^2} e^{-h\nu/k_BT} \; . \tag{5}$$

Now we see that h *does* appear in the formula; quantum mechanics is important. But does this formula resolve the ultraviolet catastrophe?

We'll go through this step by step to show how it's done. Remember that the ultraviolet catastrophe was that at high frequencies, the spectrum would diverge to infinity if the spectrum really were proportional to ν^2 for any ν. We can manipulate the equation to determine whether that is true for the full equation, in the limit that we have above.

We'll begin by noting that we are interested in varying ν for a fixed temperature T. That is, we are thinking about a blackbody of *fixed* temperature; for that blackbody, what happens at really high frequencies? Let's again use $x \equiv h\nu/k_BT$; that means that our expression becomes

$$B(\nu, T) \approx \frac{8\pi h\nu^3}{c^2} e^{-x} \ . \tag{6}$$

But $x \equiv h\nu/k_BT$ means that $\nu = k_BTx/h$, so substituting that in gives

$$B(x, T) \approx \frac{8\pi k_B^3 T^3 x^3}{h^2c^2} e^{-x} = \frac{8\pi k_B^3 T^3}{h^2c^2} x^3 e^{-x} \ . \tag{7}$$

Remembering that we are considering a constant temperature T, this expression is therefore a constant times x^3e^{-x}. As a result, we only need to determine whether as x becomes larger and larger, x^3e^{-x} becomes larger and larger, or whether it eventually decreases.

It happens that for sufficiently large x, e^{-x} decreases more rapidly with increasing x than *any* polynomial increases. But rather than just taking that as given we can get some insight by putting numbers into a calculator. If $x = 10$, then $x^3e^{-x} = 0.045$. If $x = 20$, then $x^3e^{-x} = 1.6 \times 10^{-5}$. If $x = 30$, then $x^3e^{-x} = 2.5 \times 10^{-9}$. Indeed, when x becomes large, x^3e^{-x} tends to zero and *not* infinity, like the low-frequency formula would have suggested! This is how Planck's formula saved the universe, which otherwise would have been vaporized at high frequencies :).

There are a few other points about blackbodies that are useful to keep in mind:

- At low frequencies, as we found, increasing the frequency ν of the radiation increases $B(\nu, T)$. At high frequencies, increasing the frequency of the radiation decreases $B(\nu, T)$. Thus with increasing frequency a blackbody's emission rises smoothly, reaches a peak, and then decreases smoothly.

- The peak frequency of a blackbody (i.e., the frequency ν that maximizes $B(\nu, T)$ at a given T) is proportional to T.

- For a fixed emitting area, a higher-temperature blackbody emits more at *all* frequencies than a lower-temperature blackbody. Thus even at the peak of the lower-temperature blackbody (which is at a lower frequency than the peak of the higher-temperature blackbody), the higher-temperature blackbody emits more. You can find some images on the Web that demonstrate this beautifully.

- The total flux (energy per area per time) over all frequencies of a blackbody of temperature T is $F = \sigma_{\rm SB}T^4$, where $\sigma_{\rm SB} = 5.67037 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4} is the Stefan-Boltzmann constant. Thus a higher-temperature blackbody emits a *lot* more than a lower-temperature blackbody. For example, if you double the temperature, the flux goes up by a factor of $2^4 = 16$!

- This means that if you have an object with an area A and a temperature T, the blackbody luminosity (energy per time) is $L = A\sigma_{\rm SB}T^4$.

And, as always, feel free to talk with the tutors, the TAs, or me about the topics in this supplement!

Practice problems

1. Calculate the frequency and energy of a photon of wavelength 5×10^{-7} m, which is in the middle of the visible band.

Answer: $f = c/\lambda$, so when $\lambda = 5 \times 10^{-7}$ m, putting in $c \approx 2.998 \times 10^8$ m s^{-1} gives $f = 5.996 \times 10^{14}$ Hz. The energy is then $E = hf$, so using $h \approx 6.626 \times 10^{-34}$ m^2 kg s^{-1} gives $E = 3.973 \times 10^{-19}$ J.

2. Calculate the frequency and energy of a photon of wavelength 10^{-10} m, which is in the X-ray band.

3. Calculate the linear momentum of the visible photon and the X-ray photon from the previous two problems.

4. Calculate $k_B T$ for your body temperature, $T \approx 310$ K.

Answer: $k_B \approx 1.38 \times 10^{-23}$ J K^{-1}, so $k_B T \approx 4.26 \times 10^{-21}$ J.

5. What is the wavelength of a photon of that energy?

Answer: $E = hf = hc/\lambda$, so $\lambda = hc/E$. Putting in $h = 6.626 \times 10^{-34}$ m^2 kg s^{-1}, $c = 2.998 \times 10^8$ m s^{-1}, and $E = 4.26 \times 10^{-21}$ J gives $\lambda = 4.66 \times 10^{-5}$ m.

6. Do the same calculations as in the last two problems for $T = 5,800$ K (about the value for the photosphere of the Sun).

7. Do the same calculations for $T = 10^7$ K (attainable by a neutron star).

8. The average photospheric temperature of the Sun is about $T = 5,800$ K. Compare the total blackbody flux at that temperature with the total blackbody flux from a sunspot at $T = 4,000$ K (the temperature can go down to $T = 3,000$ K in a sunspot). Why do sunspots appear dark?

9. Can you prove that the peak frequency of a blackbody is proportional to its temperature T? *Hint:* you can save yourself a lot of effort by looking carefully at the blackbody expression before trying to do any actual calculations. For example, does the factor $2h/c^3$ affect the location of the peak?

10. When we say that the peak frequency of a blackbody is proportional to T, we mean that $h\nu_{\rm peak} = Ck_B T$, where C is some numerical constant. Can you determine C to within, say, 1%? *Hint:* yes, you could solve this with calculus. But can you think of an efficient way to solve the problem using a calculator and educated guessing?