## Gravity and Its Consequences

Newton's law of universal gravitation has played a profound role in science and even in philosophy: the demonstration that such a simple law can explain both planetary orbits and the fall of an apple onto a physicist's head helped provide a sense of unity to the universe. In this supplement we will explore some of the consequences of the physics of gravity. There are a fair number of details in this supplement, and my hope is that as you follow them you will get an idea of how to perform such analyses. Here's the quick summary:

- When two point masses are in orbit around each other, the orbit is mathematically equivalent to the orbit of a single object around a fixed object at the center of mass. This means that we can analyze binary stars, where there is no single dominant mass, in the same way that we can analyze a planet around a star.
- There are many situations in astrophysics where we want to consider a slight change to a system. For example, we might ask what happens to the gravitational force if the distance is changed by a small amount. In such cases, it is extremely helpful to make use of calculus concepts.

In this supplement we will go into details about these main points, and will in particular give several examples of how to deal with small changes using calculus.

### 1. The law of gravity, the center of mass, and the reduction of the two-body problem to the one-body problem

Newton's law says that if two objects of masses $m_1$ and $m_2$ are at locations $\vec{r}_1$ and $\vec{r}_2$, then object 2 attracts object 1 with a force

$$\vec{F}_{21} = -\frac{Gm_1 m_2}{r^2}\hat{r} \tag{1}$$

where $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $r = |\vec{r}|$, and $\hat{r} \equiv \vec{r}/r$. In what ways does this satisfy the constraints on forces that we discussed two classes ago? The force on 2 due to 1 is equal and opposite to that on 1 due to 2, and the force is directed along the line between the two objects. **Note also that the force only depends on $\vec{r}_1 - \vec{r}_2$, and not the two positions separately, as is intuitively reasonable**. Newton's gravitational constant $G \approx 6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$ has units that are notoriously difficult to remember. The easiest way is to remember a formula involving $G$ (such as the force formula above!), and work it out from the known units of force, mass, and distance.

The usual approach (which we're adopting in our course as well) is to begin by thinking about a situation in which one of the masses (say, $m_1$) is much greater than the other. The Sun-Earth system is a good example; the Earth has mass, but the Sun's mass is about $3 \times 10^5$ times greater. Thus it's a pretty good approximation to say that the Sun is nailed in place while the Earth orbits around it. Using this approximation, it is possible to show that Kepler's laws of planetary motion follow from the inverse-square law of gravity. That's a great triumph.

But many orbiting systems do *not* have huge differences in masses between their components. For example, the nearest star system to our own has three stars (Alpha Centauri A, Alpha Centauri B, and Proxima Centauri). Alpha Centauri A and B form a binary with an orbital period of 80 years (Proxima is much farther away). Alpha Centauri A has a mass 1.1 times the mass of our Sun; Alpha Centauri B has a mass 0.9 times the mass of our Sun. Clearly, in that system, we can't assume that one star is fixed while the other orbits!

At first sight, this might appear to be a huge problem. What seems simple for a planet orbiting a star could in principle become hugely complicated for two stars orbiting each other.

But luckily, in Newtonian gravity, we can show that the problem of the orbit of two point masses around each other reduces to the problem of one body around the unmoving center of mass! That makes things a lot easier.

Let's first convince ourselves that the center of mass is the right point. You've probably all played on a seesaw, and you know that if you are sitting across from a 500 pound gorilla then the gorilla will need to sit much closer to the fulcrum than you do, for there to be a balance between the two of you. In fact, if you and your gorilla friend want a perfect balance, then you need to adjust so that the fulcrum is at the balance point.

Put in more abstract terms, let objects of masses $m_1$ and $m_2$ be, respectively, at the locations $\vec{r}_1$ and $\vec{r}_2$. Then the center of mass is at

$$\vec{r} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \; . \tag{2}$$

Of course, the system might be moving with respect to us. Whether it is or not, the total momentum of the system is $\vec{p}_{\rm tot} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = (m_1 + m_2)\dot{\vec{r}} = M_{\rm tot}\dot{\vec{r}}$, where $M_{\rm tot} \equiv m_1 + m_2$ is the total mass of the system. Therefore, the center of mass is also the center of momentum.

We will now show that we can turn a two-body problem (in which we think about the forces on each object separately) into a one-body problem (in which we imagine a single body orbiting the center of mass). We'll start by defining the position of each object with respect to the center of mass as $\vec{R}_1 \equiv \vec{r}_1 - \vec{r}$ and $\vec{R}_2 \equiv \vec{r}_2 - \vec{r}$, and the relative positions of the two masses as $\vec{R} \equiv \vec{R}_1 - \vec{R}_2 = \vec{r}_1 - \vec{r}_2$. The $\vec{F} = m\vec{a}$ equations of motion are then

$$\begin{aligned} m_1\ddot{\vec{r}}_1 &= -\tfrac{Gm_1m_2}{R^3}\vec{R} \\ m_2\ddot{\vec{r}}_2 &= \tfrac{Gm_1m_2}{R^3}\vec{R} \end{aligned} \tag{3}$$

(remember that one dot means a single time derivative, and two dots means two time derivatives; also remember that $\vec{R}/R = \hat{R}$, so these equations are the same as Equation 1). Adding these together we get

$$\begin{aligned} m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 &= 0 \\ (m_1+m_2)\ddot{\vec{r}} &= 0 \\ (m_1+m_2)\dot{\vec{r}} &= \text{constant} . \end{aligned} \tag{4}$$

In the last line we use the constancy of the masses; you can take the derivatives of both sides to get the previous line. What does this mean? It means that the total momentum $\vec{p}_{\rm tot} = (m_1+m_2)\dot{\vec{r}}$ is constant, which it had better be! It is useful to do these types of checks on occasion during a derivation.

Now let's multiply the first of our equations of motion by $m_2$, the second by $m_1$, and subtract:

$$m_1m_2(\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = -\frac{Gm_1m_2(m_1+m_2)}{R^3}\vec{R} . \tag{5}$$

We recognize that because $\vec{R} = \vec{R}_1 - \vec{R}_2 = \vec{r}_1 - \vec{r}_2$, the expression in parentheses on the left hand side is just $\ddot{\vec{R}}$, so after cancelling the product $m_1m_2$ on both sides we have finally

$$\ddot{\vec{R}} = -\frac{G(m_1+m_2)}{R^3}\vec{R} . \tag{6}$$

This equation means that the two-body problem reduces *exactly* to the one-body problem, except that the mass is now the total mass and the vector $\vec{R}$ that is changing doesn't represent the actual position of a body, but rather the separation vector of the two bodies. Wow! This is pretty cool, because it means that we can now transfer all the insight we gained in one-body orbits to two-body orbits.

Note that the relative motion of the objects is *independent* of the initial coordinate system we used (the one in which the positions of the bodies are $\vec{r}_1$ and $\vec{r}_2$). This has to be the case; it's an example of a symmetry. If it were otherwise, then, for example, the orbits of planets in the Solar System would depend on which alien happened to be observing us at a given time!

What if we want the motion of each individual body? First, we solve the equivalent one-body problem for $\vec{R}$. We then use

$$\begin{aligned} m_1\vec{r}_1 + m_2\vec{r}_2 &= (m_1+m_2)\vec{r} \\ \vec{r}_1 - \vec{r}_2 &= \vec{R} . \end{aligned} \tag{7}$$

How can we solve for $\vec{r}_1$ and $\vec{r}_2$ independently? A good way to start would be to multiply the second equation by $m_2$ to get

$$m_2\vec{r}_1 - m_2\vec{r}_2 = m_2\vec{R} . \tag{8}$$

When we add this to the first equation, the $m_2\vec{r}_2$ terms cancel out, and we therefore get

$$(m_1+m_2)\vec{r}_1 = (m_1+m_2)\vec{r} + m_2\vec{R} , \tag{9}$$

and so on. The final result is

$$\begin{aligned} \vec{r}_1 &= \vec{r} + m_2/(m_1+m_2)\vec{R} = \vec{r} + \tfrac{\mu}{m_1}\vec{R} \\ \vec{r}_2 &= \vec{r} - m_1/(m_1+m_2)\vec{R} = \vec{r} - \tfrac{\mu}{m_2}\vec{R} \end{aligned} \tag{10}$$

where we have defined the *reduced mass* $\mu \equiv m_1 m_2/(m_1 + m_2)$. If you look at the motions of the two bodies in detail, you find that each of them moves in an ellipse with one focus being at the center of mass of the system.

## 2. Some consequences for orbits

Because of this reduction from a two-body system to a one-body system, we can revisit some aspects of orbits, in particular their energy and angular momentum.

Suppose that we have two objects, of masses $m_1$ and $m_2$, which orbit around each other in an ellipse with semimajor axis $a$ and eccentricity $e$. Here, we are to think about the relative separation between the objects, which ranges from $a(1-e)$ at the closest to $a(1+e)$ at the farthest. Again define $M_{\rm tot} \equiv m_1 + m_2$ to be the total mass, and $\mu \equiv m_1 m_2/(m_1 + m_2)$ to be the reduced mass. Then the total orbital energy is

$$E_{\rm orb} = -\frac{GM_{\rm tot}\mu}{2a} \tag{11}$$

and the total orbital angular momentum is

$$L_{\rm orb} = \mu\sqrt{GM_{\rm tot}a(1-e^2)}\ . \tag{12}$$

Note that these expressions are, correctly, unchanged if we decide that we will rename $m_1$ to $m_2$, and vice versa: since neither $M_{\rm tot}$ nor $\mu$ change in that case, and $a$ and $e$ are independent of what we're calling the masses, then the energy and angular momentum don't change; how could they!

We see as before that the energy does not depend on the eccentricity, just the semimajor axis. The angular momentum, however, does depend on the eccentricity, as it must: $e \to 1$ is the limit toward a purely radial orbit, which has zero angular momentum at fixed $a$, and the expression indicates that correctly.

Finally, we note that in the limit where one object has a much smaller mass than the other (say, $m_2 \ll m_1$), the expressions go back to what we're used to for (say) a planet around the star. In the $m_2 \ll m_1$ limit, $M_{\rm tot} \approx m_1$ and $\mu \approx m_2$ (work it out), and then the energy and angular momentum reduce to what we had previously.

## 3. Gravitational potential energy, and using calculus to approximate

Let's now temporarily retreat back to the case in which we have a low-mass object orbiting around a much more massive object. We know that the total energy of the system has to be conserved if the system is isolated. We already know the formula for the kinetic energy: for something of mass $m$ and speed $v$, the kinetic energy is $E_{\rm kin} = \frac{1}{2}mv^2$.

But in addition, there is gravitational *potential energy* in the system. If at a given moment masses $m_1$ and $m_2$ are separated by a distance $r$, then the gravitational potential energy between the two masses is

$$E_{\rm pot} = -\frac{Gm_1m_2}{r} \ . \tag{13}$$

A question that might strike us is: why is the energy *negative*? We can answer this with two points:

1. The energy scale is by convention set so that two bodies at infinite distance from each other have zero gravitational potential energy. I say "by convention" because it turns out that it is energy *differences* that matter rather than the absolute scale of energies. It is just convenient in many applications to set the gravitational potential to zero at infinite separation.

2. Using that scale, we can convince ourselves that closer bodies must have more negative gravitational potential energy. Remember that gravitation is universally attractive. Thus if we start with two objects at some finite distance from each other, to get them to infinite distance we have to pull them apart. That takes positive energy (in the sense of physics, not of woo-woo New Age nonsense!). Since you need to add energy to get to zero, the original energy must have been negative.

You have probably seen gravitational potential energy in a different form. For example, you might have seen gravitational potential energy in the form

$$E_{\rm pot} = mgh \tag{14}$$

for an object of mass $m$ a height $h$ above the ground, when the gravitational acceleration is $g$. This looks very different from our formula above, so what gives?

First, note that here we have an example of another scale of energy: the potential energy is 0 at the ground, rather than at an infinite distance. As we said above, that's no problem, because it is differences in energy rather than the absolute energy value that matter.

But the much different look of this formula from our previous one deserves some scrutiny. Say that we are dealing with a mass $m$ on the Earth, which has a mass $M \gg m$. If we are at a distance $r$ from the center of the Earth (which we'll treat as spherical), then the magnitude of the gravitational force is $F = GMm/r^2$ (no vector symbols here, because we're only thinking about the magnitude). Thus $F = ma$ tells us that the gravitational acceleration is $g = F/m = GM/r^2$. Our second formula for the gravitational potential energy then becomes

$$E_{\rm pot} = \frac{GMm}{r^2} h \ . \tag{15}$$

That's still not the same, though. Let's say that we use our first formula, $E_{\rm pot} = -GMm/r$, and ask about the potential energy *difference* between a mass $m$ at a distance $r$ from the center of our spherical object of mass $M$, and the same mass $m$ at a distance $r + h$ from the center. Using $\Delta$ to represent the difference, we get

$$\Delta E_{\rm pot} = E_{\rm pot}(r+h) - E_{\rm pot}(r) = -\frac{GMm}{r+h} - \left(-\frac{GMm}{r}\right) \ . \tag{16}$$

This still doesn't seem to help us a lot. But let's make one additional assumption: that $h \ll r$. Then, it will turn out that we can use calculus.

The way we use calculus is to start from the definition of a derivative. If we have a function $f(x)$, then

$$\frac{df}{dx} = \lim_{dx\rightarrow 0} \frac{f(x+dx) - f(x)}{dx} \ . \tag{17}$$

This means that for a small but nonzero $dx$, we can multiply both sides by $dx$ to get

$$f(x+dx) - f(x) \approx dx\frac{df}{dx} \ . \tag{18}$$

For our case, we can consider the derivative of the gravitational potential energy with $r$:

$$E_{\rm pot}(r+dr) - E_{\rm pot}(r) \approx dr\frac{dE_{\rm pot}}{dr} \ . \tag{19}$$

But the radius change we are considering is $dr = h$, so

$$E_{\rm pot}(r+h) - E_{\rm pot}(r) \approx h\frac{dE_{\rm pot}}{dr} \ . \tag{20}$$

Please remember that this is an *approximation*, which becomes closer and closer to true as $h$ becomes smaller and smaller than $r$. For something like the Earth, where maybe in a lab experiment $h = 1$ meter and $r > 6,000$ km, indeed $h \ll r$. Because $E_{\rm pot} = -GMm/r$, $dE_{\rm pot}/dr = GMm/r^2$, and therefore the difference in potential energy is

$$E_{\rm pot}(r+h) - E_{\rm pot}(r) \approx hGMm/r^2 = mgh \ . \tag{21}$$

Ta da! The $mgh$ formula is true in the limit of the more general formula, that the change in height is much less than the original radius.

But the technique here is one that you should remember. If you have any function $f(x)$, and you are interested in the change in $f$ (i.e., $\Delta f$) from $x$ to $x + \Delta x$, then in the limit that $\Delta x$ is very small,

$$\Delta f \approx \Delta x\frac{df}{dx} \ . \tag{22}$$

## 4. Tidal force

As one more important application of Newton's law of gravity, and as another illustration of our calculus approximation approach, we will consider tidal forces. These are indeed named after ocean tides, which are caused by the gravity of the Moon and the Sun. But how exactly do they work?

Some initial thinking should convince us that "gravity" by itself is not a sufficient explanation for the tides. You know, for example, that astronauts in the International Space Station feel weightless; even though gravity *is* operating on them, they fall freely and thus don't *feel* the "pull" of gravity. Similarly, when you jump off of a diving board you don't feel the pull of gravity when you are in the air. **The observation that if you are freely falling you don't feel gravity, is actually one statement of a profound physical principle called the** ***equivalence principle*****.** We'll get to this in much more detail in ASTR 121, when we talk about general relativity.

Thus something else must cause tides. A clue we can get is by looking again at Equation 1. The acceleration of gravity of course depends on the distance from the source of gravity. But a planet such as Earth is big; its near part is closer to the Moon than its center, and its center is closer to the Moon than its far part. This suggests that the near part will be pulled more toward the Moon than the center will be, and that the center will be pulled more toward the Moon than the far part. This is what causes tides. There is an extra bulge on the near part of the Earth due to this effect, *and* an extra bulge on the far part of the Earth; you can think of the near part of the Earth being pulled away from the center of the Earth, and of the center of the Earth being pulled away from the far part. That's why we get *two* tides per day and not just one.

But how strong is the tidal effect? Let's think about the Moon-Earth situation. Say that the distance from the center of the Earth to the Moon is $r$, and that the radius of the Earth is $R$. Let the mass of the Earth be $M_E$, and of the Moon be $M_M$. Then the magnitude of the force at a distance $r$ is

$$F(r) = \frac{GM_E M_M}{r^2} \ , \tag{23}$$

and the magnitude of the force at a distance $r - R$ (i.e., the near side) is

$$F(r-R) = \frac{GM_E M_M}{(r-R)^2} \ . \tag{24}$$

We want the *difference* in the forces, $F(r-R) - F(r)$. In general, this would be a complicated problem. But the radius of the Earth is about 1/60 of the distance to the Moon, so $R \ll r$ and we can use our calculus trick:

$$F(r-R) - F(r) = \frac{GM_E M_M}{(r-R)^2} - \frac{GM_E M_M}{r^2} \approx (-R)\frac{dF}{dr} = R\frac{2GM_E M_M}{r^3} = \frac{2RGM_E M_M}{r^3} \ . \tag{25}$$

Note, by the way, that you get the same answer if you compare the tidal force between the center of the Earth (at distance $r$) and the far side of the Earth (at distance $r + R$); try it!

We can now do a quick comparison of the ratio of the tidal force exerted on the Earth by the Moon, to the tidal force exerted on the Earth by the Sun. The Sun is far more massive, but it is also much farther away than the Moon. Which wins?

If we use $r_{\rm Moon}$ to denote the distance to the Moon, $r_{\rm Sun}$ to denote the distance to the Sun, and $M_S$ to denote the mass of the Sun, then the Moon-Earth tidal force is

$$F_{\rm tide,Moon} = \frac{2RGM_E M_M}{r^3_{\rm Moon}} \ , \tag{26}$$

where again $R$ is the radius of the Earth, and the Sun-Earth tidal force is

$$F_{\rm tide,Sun} = \frac{2RGM_EM_S}{r_{\rm Sun}^3} \ . \tag{27}$$

We want the ratio of these forces. An *inefficient* way to compute the ratio would be to plug in all the numbers for $F_{\rm tide,Moon}$, then plug in all the numbers for $F_{\rm tide,Sun}$, then take the ratio. You would be very likely to make a mistake somewhere. An *efficient* way to compute the ratio is to cancel out common factors first. We see that $2RGM_E$ is a common factor, so we cancel that out. Then we have

$$\frac{F_{\rm tide,Moon}}{F_{\rm tide,Earth}} = \frac{M_M/r_{\rm Moon}^3}{M_s/r_{\rm Sun}^3} = \frac{M_M}{M_S}\left(\frac{r_{\rm Sun}}{r_{\rm Moon}}\right)^3 \ . \tag{28}$$

*Now* we plug in the numbers. $M_M/M_S = 7.35 \times 10^{22}$ kg$/1.989 \times 10^{30}$ kg $= 3.70 \times 10^{-8}$, and $r_{\rm Sun}/r_{\rm Moon} = 1.496 \times 10^{11}$ m$/3.844 \times 10^8$ m $= 389$. Thus

$$\frac{F_{\rm tide,Moon}}{F_{\rm tide,Earth}} = 3.70 \times 10^{-8} \times (389)^3 = 2.18 \ . \tag{29}$$

Despite its much smaller mass, the closeness of the Moon means that it exerts about twice the tidal force on Earth that the Sun does.

And, as always, feel free to talk with the tutors, the TAs, or me about the topics in this supplement!

# Practice problems

1. On average the Earth is $r = 1.496 \times 10^{11}$ m from the Sun. The Earth's mass is $M_\oplus = 5.972 \times 10^{24}$ kg, and the Sun's mass is $M_\odot = 1.989 \times 10^{30}$ kg. In Newtons (standard SI unit), what is the magnitude of the force of gravity between them? Recall that in SI units, $G = 6.67 \times 10^{-11}$ m$^3$ kg$^{-1}$ s$^{-2}$.

**Answer:** the magnitude of the force of gravity between two masses $m_1$ and $m_2$ separated by a distance $r$ is $F = Gm_1m_2/r^2$. Plugging in the numbers gives $F = 3.54 \times 10^{22}$ Newtons.

2. Do the same calculation as above for Venus ($M_{\rm Venus} = 4.871 \times 10^{24}$ kg, $r = 1.082 \times 10^{11}$ m).

3. Two friends, one of mass 60 kg and one of mass 90 kg, shake hands. At that moment, their centers of gravity are 0.5 m from each other. Treating them both as spherical (we're astrophysicists, after all!), calculate the force of gravity between them during their handshake. Compare that with the weight of a typical bacterium (mass $10^{-15}$ kg) on the surface of the Earth (mass $M_\oplus = 5.972 \times 10^{24}$ kg, radius $R_\oplus = 6.378 \times 10^6$ m).

4. Suppose you have a circular disk. Figure out the area of the part of the disk between a radius $r$ and a radius $r + \Delta r$, where $\Delta r \ll r$.

**Answer:** The area of a disk of radius $r$ is $\pi r^2$, and the area of a disk of radius $r + \Delta r$ is $\pi(r + \Delta r)^2$. The difference is

$$\begin{aligned} dA &= \pi(r + \Delta r)^2 - \pi r^2 \\ &= \pi[r^2 + 2r\Delta r + (\Delta r)^2 - r^2] \\ &= \pi[2r\Delta r + (\Delta r)^2] \\ &\approx \pi(2r\Delta r)\ . \end{aligned} \tag{30}$$

In the last step, we note that because $\Delta r \ll r$, $(\Delta r)^2 \ll 2r\Delta r$, so we drop the $(\Delta r)^2$ term.

We can also do this with calculus:

$$\begin{aligned} A &= \pi r^2 \\ dA/dr &= 2\pi r \\ \Delta A &= \Delta r(dA/dr) \\ \Delta A &= 2\pi r\Delta r\ , \end{aligned} \tag{31}$$

in the limit $\Delta r \ll r$.

5. Suppose you have a sphere. Figure out the volume of the part of the sphere between a radius $r$ and a radius $r + \Delta r$, where $\Delta r \ll r$.

6. Remember that the kinetic energy of an object of mass $m$ moving at speed $v$ is $E_{\rm kin} = \frac{1}{2}mv^2$. How much energy do you need to put in to the object to increase its speed to $v + dv$, if $dv \ll v$?

7. Some advocates for astrology, when pressed to come up with a physical mechanism by which astronomical bodies can influence human beings, say "tidal forces". A common argument along

those lines is that if the Moon can produce tides on the huge Earth, surely it can have a big effect on us. Calculate that effect. Suppose we consider a 50 cm long newborn whose center is 2 meters from a 70 kg doctor. Compare the tidal force from the doctor on the baby, with the tidal force of the Moon on the baby. Which is greater? Also, figure out the ratio of the true tidal force of the doctor on the baby, to the tidal force we get using our calculus-based approximation. What fractional error do we make by using the approximation?