Conservation Laws

Suppose that two spaceships collide. The aftermath has twisted wreckage, fuel leaks, and other awful things. But is there something about the after that is the same as the before?

The answer is yes. Conserved quantities (ones that do not change after some interaction) are at the heart of much of the intuition we have about physics, and about many related astronomy topics. In this supplement we will discuss some conservation laws that follow from Newton's laws of motion, and that are indeed even more general.

But first we should address an issue that can cause confusion. When we talk about conserved quantities such as energy, linear momentum, and angular momentum, we have in mind *isolated* systems. That is, we imagine that we have a system that is completely isolated from any contact with anything else, and we include everything about that isolated system, before and after the interaction. Thus, for example, if we are thinking about a star as part of our system, we would track every photon emitted by that star and include it in our calculations. If we don't do that, then the remaining parts of the system *can* change their energy, linear momentum, angular momentum, or other quantities.

This could lead to some reasonable objections. *No* system in the universe is isolated in this sense. For example, we receive radiation from the microwave background, so clearly we get energy from afar; we also radiate energy, stuff hits the Earth that can bring linear or angular momentum, and so on. Thus the assumption of an isolated system sounds absurdly unrealistic. Why should we travel in such a theoretical dreamland?

The answer is that many systems are *almost* isolated. For example, consider a binary consisting of two ordinary stars. Yes, that system will radiate energy, angular momentum, and so on, but the time needed to radiate a significant fraction of the system's angular momentum or energy is extremely large compared with a binary orbital period. Thus over a binary period, energy and angular momentum are almost conserved.

On a more fundamental level, we are talking about basic physical laws, and we get improved insight about them when we think about idealized situations. You know, for example, that a rock and a piece of paper don't fall in the same way on Earth, but *if* you eliminated air resistance, they would. In the same way, we can show that *if* you included absolutely everything: every atom, photon, neutrino, etc., then there are quantities that remain constant. That's the spirit in which we will investigate conserved quantities. And again, lots of systems are basically isolated, so even locally and without including the whole universe, these laws are often obeyed to high precision.

With that in mind, let's proceed!

1. Conservation of Linear Momentum

Linear momentum, often just called "momentum", is $\vec{p} = m\vec{v}$ for slowly-moving systems. It is obvious that the linear momentum of a single, non-isolated object can change. For example, the Earth orbits the Sun; thus six months from now, it will be moving in the opposite direction from how it is moving now. But what if we think about the linear momentum of the Earth *plus* the Sun?

We will start by imagining an isolated two-body system. We only have two particles in the system, A and B. The force of A on B is $\vec{F}_{AB}$, and the force of B on A is $\vec{F}_{BA}$. There are no other forces exerted on either A or B. We'd like to examine the rate of change of the total momentum in such a system. Why? Because, as we indicated in the last supplement, if we find that the rate of change of the momentum is zero, then the momentum must be constant because it isn't changing.

From Newton's second law, the rate of change of the momentum $\vec{p}_A$ of particle A is

$$\frac{d\vec{p}_A}{dt} = \vec{F}_{BA} \; , \tag{1}$$

and similarly the rate of change of the momentum $\vec{p}_B$ of particle B is

$$\frac{d\vec{p}_B}{dt} = \vec{F}_{AB} \; . \tag{2}$$

The momentum of the *full* system (which contains just A and B) is $\vec{p}_{\rm tot} = \vec{p}_A + \vec{p}_B$. Its rate of change is

$$\frac{d\vec{p}_{\rm tot}}{dt} = \frac{d(\vec{p}_A + \vec{p}_B)}{dt} = \frac{d\vec{p}_A}{dt} + \frac{d\vec{p}_B}{dt} = \vec{F}_{BA} + \vec{F}_{AB} \; . \tag{3}$$

But Newton's third law says that $\vec{F}_{BA} = -\vec{F}_{AB}$, which means that $\vec{F}_{BA} + \vec{F}_{AB} = 0$. Therefore,

$$\frac{d\vec{p}_{\rm tot}}{dt} = 0 \; . \tag{4}$$

The total momentum of the system is constant, i.e., the total momentum of the system is conserved.

That sounds promising, but it might initially seem that all we've done is show that momentum is conserved if we have exactly two particles. But it's much more general than that. Suppose, for example, that we have some potentially large number of particles A, B, C, D, ... The argument above tells us that particle A's *contribution* to the change in momentum of particle B is exactly cancelled by particle B's contribution to the change in momentum of particle A. This cancellation works for *every* pair of particles: AC cancels CA, BD cancels DB, and so on. Thus momentum is conserved for the entire system!

Note again that we do have to include the entire system for this to be true. For example, as we mentioned before, if we look just at the Earth we notice that its momentum clearly changes over its orbit. But if the Earth and Sun formed an isolated system then the total momentum of the Earth-Sun system would be conserved.

2. Conservation of Angular Momentum

For our investigation of the conservation of angular momentum $\vec{L} = \vec{r} \times \vec{p}$, we will again start with a two-particle system. We will also add one more fact: that the force between two particles acts along the line between them (anything else would violate symmetry; think about it!). Thus, for example, if particle A is at vector location $\vec{r}_A$ in some coordinate system, and particle B is at vector location $\vec{r}_B$, then the force between them has a *direction* that is parallel or antiparallel to $\vec{r}_B - \vec{r}_A$. The force can go inversely with distance (as in gravity), or have some other dependence, and could be attractive (as with gravity) or repulsive (as with two electrons, due to the electrostatic force), depending on the nature of the force.

With this in mind, how does the angular momentum evolve for an isolated two-particle system? The time rate of change of the total angular momentum $\vec{L}_{\rm tot} \equiv \vec{L}_A + \vec{L}_B$ is

$$\begin{aligned} d\vec{L}_{\rm tot}/dt &= \vec{r}_A \times \vec{F}_{BA} + \vec{r}_B \times \vec{F}_{AB} \\ &= -\vec{r}_A \times \vec{F}_{AB} + \vec{r}_B \times \vec{F}_{AB} \\ &= (\vec{r}_B - \vec{r}_A) \times \vec{F}_{AB} \\ &\propto (\vec{r}_B - \vec{r}_A) \times (\vec{r}_B - \vec{r}_A) \\ &= 0\ . \end{aligned} \tag{5}$$

In the last line we used the property of cross products that the cross product between any two parallel, or antiparallel, vectors is zero.

Therefore, for an isolated two-particle system, the angular momentum is conserved. What happens for an isolated system with more than two particles? As with linear momentum, the angular momentum is still constant, as can be seen from pairwise cancellation. Thus, *for any isolated system, the linear momentum and angular momentum are constant.* Another way of putting this is that for *any* system, isolated or otherwise, the total linear and angular momentum are changed only by *external* forces and torques, respectively:

$$d\vec{p}_{\rm tot}/dt = \vec{F}_{\rm ext}, \qquad d\vec{L}_{\rm tot}/dt = \vec{N}_{\rm ext}\ . \tag{6}$$

3. Conservation of Energy

We now move on to a third conservation principle: the conservation of energy. Unlike linear and angular momentum, energy is tricky to define rigorously. One try might be "Energy is a quantity which may be converted into motion". If all you have to worry about is energy of motion, it's straightforward: $E = \frac{1}{2}mv^2$ for nonrelativistic motion. However, there are many other forms of energy: potential energy, electrostatic energy, thermal energy, chemical energy, nuclear energy, energy in photons or neutrinos or gravitational waves, and so on. Not all of those forms are distinct at a fundamental level, but it doesn't matter. In a general case, it can be difficult to track where all the energy goes, but the principle of energy conservation is that for an isolated system (as always), the total energy in all forms is constant, although the amount of energy in each form can change.

Richard Feynman used a nice analogy for this, which I'll paraphrase closely. Suppose a child receives a toy for Christmas: 28 indestructible blocks. He plays with them in his room. He's rather messy, so his father comes in to clean up every once in a while. After a while, the father notices an amazing thing: day after day, there are always 28 blocks in the room! One day, there are only 27 visible; however, a search reveals that one of the blocks is under the rug, so the total is still 28. Another day there are only 26. Careful examination shows that the window is open, and indeed the two missing blocks are outside. Another day there are 30 blocks! Yikes! But it turns out that the child had a friend visiting, and the friend brought two blocks with them, so that's okay. The next day brought a puzzle; only 25 blocks are visible, and the father has searched everywhere but the toy box, which is closed. The child throws a tantrum and refuses to allow the box to be opened. Being clever, however, the father weighs the toy box and discovers that it has excess weight exactly equal to three blocks. On yet another day, only 20 blocks can be seen. After all the other possibilities have been eliminated, the father notes that the bath water is higher than it was when the water was poured in. The bath water is dirty, so the father can't directly check if the blocks are in there, but using Archimedes' principle he finds that just the right amount of water is displaced for 8 blocks. As time goes on, in fact, he discovers that there are always 28 blocks, although creativity may be required to discover where they are. Just as with the blocks, the total energy in a system is always conserved, but sometimes it takes some creativity to determine where it has gone.

For additional research: if you are interested in conservation laws, you might want to look up Noether's Theorem. This is a beautiful theorem in mathematical physics proven by the great Emmy Noether in 1915. It shows that *symmetries* of a particular type are inevitably linked to *conserved quantities*. In our case, it turns out that time symmetry (doing a local experiment now in an isolated laboratory gives the same result as doing it earlier or later) gives us energy conservation, and similarly location symmetry (doing an isolated experiment here or there) leads to linear momentum conservation, and angular symmetry (orienting an isolated laboratory to this angle or that) implies angular momentum conservation.

And, as always, feel free to talk with us about the topics in this supplement!

Practice problems

1. A ball of mass $m = 2$ kg is 1 m above a table on Earth, which has a gravitational acceleration $g \approx 9.8$ m s^{-2}. If it is dropped from there, with zero initial speed, and if we ignore air resistance and other complicating effects, how fast will it be moving when it hits the table?

Answer: The gravitational potential energy relative to the table is $E_{\rm pot} = mgh$. Here h is the height relative to the table, so $h = 0$ when the ball hits the table. The kinetic energy of motion is $E_{\rm kin} = \frac{1}{2}mv^2$, where v is the speed. When the ball hits the table, $h = 0$ and thus $E_{\rm pot} = 0$. The sum of the energies, $E_{\rm pot} + E_{\rm kin}$, is always constant, and when the ball hits the table this means

$$\begin{aligned} E_{\rm kin} = \frac{1}{2}mv^2 &= mgh \\ v &= \sqrt{2gh} \\ &= \sqrt{2 \times (9.8 \text{ m s}^{-2}) \times (1 \text{ m})} \\ &\approx 4.4 \text{ m s}^{-1} . \end{aligned} \tag{7}$$

Note that the mass of the ball cancels out.

2. Suppose that the ball in the previous problem is released from 1 m, but when it is released it is pushed so that at 1 m above the table it is already moving downward at 1 m s^{-1}. How fast does it hit the table in this case?

3. A 100 kg person runs at 10 m s^{-1} head-on into a person with a mass of 150 kg. How fast does the second person have to be running in the opposite direction so that they have the same magnitude of momentum, i.e., that their total momentum is zero?

4. A block of mass 4 kg slides at 1 m s^{-1} along a surface. We will ignore both friction and air resistance, and will assume that the surface is flat. The block then hits an initially stationary block of mass 1 kg. The motion is all in a straight line; the collision is not glancing, no rotation occurs, etc. Also, the masses of the blocks are unchanged during the whole process. From conservation of energy and linear momentum, determine the speed of each block after the collision. We assume that the energy is only in the form of the kinetic energy of the blocks: the blocks don't vibrate, or heat up, or anything like that.

Answer: I'm going to go through this solution in considerable detail because, in my opinion, it displays a lot of characteristics of an astrophysical calculation. I hope it helps!

Back to the problem: because all of the motion is in a straight line, we will use the scalar versions of the formula, e.g., linear momentum is really $\vec{p} = m\vec{v}$, but we'll write it as $p = mv$. Using "1" to refer to the first block and "2" to refer to the second block, the initial total linear momentum is $p = p_1 + p_2 = m_{1,i}v_{1,i} + m_{2,i}v_{2,i} = (4 \text{ kg})(1 \text{ m s}^{-1}) + (1 \text{ kg})(0 \text{ m s}^{-1}) = 4$ kg m s^{-1}. Here the subscript "i" means "initial". The final linear momentum, after the interaction, has to be the same as the initial linear momentum, so it also must be that $m_{1,f}v_{1,f} + m_{2,f}v_{2,f} = 4$ kg m s^{-1} (where "f" means "final"). But we can save ourselves some time, and be more general, if we keep the expressions symbolic. We can also remove the "i" and "f" from the masses, because we assume that the masses stay constant.

With all that in mind, we find that

$$m_1 v_{1,f} + m_2 v_{2,f} = m_1 v_{1,i} \ . \tag{8}$$

Dividing through by m_1 gives us

$$v_{1,f} + (m_2/m_1) v_{2,f} = v_{1,i} \ . \tag{9}$$

So far we don't have enough information to find the solution. But we are also told that the total *energy* is the same after as before the collision. That is,

$$\begin{aligned} \tfrac{1}{2} m_1 v_{1,f}^2 + \tfrac{1}{2} m_2 v_{2,f}^2 &= \tfrac{1}{2} m_1 v_{1,i}^2 \\ v_{1,f}^2 + (m_2/m_1) v_{2,f}^2 &= v_{1,i}^2 \ . \end{aligned} \tag{10}$$

We can square equation (9), which gives us

$$v_{1,f}^2 + 2(m_2/m_1) v_{1,f} v_{2,f} + (m_2/m_1)^2 v_{2,f}^2 = v_{1,i}^2 \ . \tag{11}$$

Setting the two expressions for $v_{1,i}^2$ equal to each other gives

$$v_{1,f}^2 + 2(m_2/m_1) v_{1,f} v_{2,f} + (m_2/m_1)^2 v_{2,f}^2 = v_{1,f}^2 + (m_2/m_1) v_{2,f}^2 \ . \tag{12}$$

We cancel the $v_{1,f}^2$ on both sides and move a term over to get

$$2(m_2/m_1) v_{1,f} v_{2,f} + [(m_2/m_1)^2 - (m_2/m_1)] v_{2,f}^2 = 0 \ . \tag{13}$$

We *always* want to look at our expressions to see what they imply, and to determine whether that makes sense. We see, for example, that $v_{2,f} = 0$ solves this equation. What does that mean? We know that $v_{2,i}$ (the initial speed of block 2) is zero. Thus, yes, it does make sense that $v_{2,f} = 0$ is a solution, because it would effectively be the same as the initial conditions, which of course have the same linear momentum and energy as the initial conditions!

We also see that the expression only depends on the *ratio* of the masses, rather than both masses separately. Does that make sense?

If we now assume that $v_{2,f}$ is *not* zero, we can cancel it (and the common factor m_2/m_1) in Equation 13 to get

$$2 v_{1,f} + [(m_2/m_1) - 1] v_{2,f} = 0 \ , \tag{14}$$

or after some more manipulation,

$$v_{1,f} = \frac{m_1 - m_2}{2 m_1} v_{2,f} \ . \tag{15}$$

Here, again, we should stop to think. This says that, given that initially the second block isn't moving, the proportionality between the final speed of the first block and the final speed of the section block does not depend on the initial speed of the first block. Does that make sense? Also, we see that when $m_1 = m_2$, $v_{1,f} = 0$. Does *that* make sense?

Now we substitute this back into our linear momentum conservation equation, which was

$$m_1 v_{1,f} + m_2 v_{2,f} = m_1 v_{1,i} \ . \tag{16}$$

We know that $m_1 v_{1,f} = \frac{1}{2}(m_1 - m_2) v_{2,f}$, so our linear momentum conservation equation becomes

$$\begin{aligned} \tfrac{1}{2}(m_1 - m_2) v_{2,f} + m_2 v_{2,f} &= m_1 v_{1,i} \\ \tfrac{1}{2}(m_1 + m_2) v_{2,f} &= m_1 v_{1,i} \\ v_{2,f} &= \tfrac{2m_1}{m_1 + m_2} v_{1,i} \ . \end{aligned} \tag{17}$$

This also implies that

$$v_{1,f} = \frac{m_1 - m_2}{2m_1} v_{2,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i} \ . \tag{18}$$

Finally, plugging in $m_1 = 4$ kg, $m_2 = 1$ kg, and $v_{1,i} = 1$ m s^{-1} gives $v_{1,f} = (3/5)$ m s^{-1} and $v_{2,f} = (8/5)$ m s^{-1}.

So what do we learn from all this? This is a good example of an astrophysical problem. We take a simplified problem, determine what physics is important, set out the equations in an organized way, and solve them. **It is extremely important to check your expressions along the way.** By "check your expressions" we do not mean "stare carefully at your derivation" :). We mean what we did here: we ask "does this make sense?" at a number of parts of the derivation. Please do this in your own work. Remember the carrot and stick: if you get a wrong answer but correctly say why (e.g., via a check such as this) and roughly what the right answer is, you will get substantial partial credit. If you get a clearly wrong answer (wrong units, or symmetries, or limits) and do *not* say anything, you will have extra points taken off. You have to commit, though: saying "I think this might be wrong" gets you no credit. The reason I feel so strongly about this is that this type of checking ("does it make sense?") is critical to astrophysics.

5. Now we'll go through some limits of this expression, to determine whether our answer makes sense. First, suppose that $m_1 \gg m_2$. In that limit, calculate $v_{1,f}$ and $v_{2,f}$ relative to $v_{1,i}$. Verify explicitly that these conserve energy and angular momentum. Do the answers make physical sense?

6. Now the opposite limit: $m_1 \ll m_2$. Do the answers make physical sense?

7. Explicitly work through the conservation of linear momentum and angular momentum for three particles. That is, write the time rate of change of the total linear momentum, and total angular momentum, for a three-particle system, and show that the rate of change is zero.