Using Calculus to Describe Motion

Calculus and vectors are extremely powerful tools to describe motion as well as many other concepts in physics. In this supplement, we will explore motion and in particular Newton's laws of motion in this way. Please think of questions you have about these topics so that we can discuss them in class.

1. Position, velocity, acceleration, and momentum

Let's say we are interested in the position and motion of an object. We'll imagine that the object is a point, so that its position and motion can be defined uniquely; for an object that is not a point, we could imagine following a point within the object, such as its center of mass.

The *position* of the object can be specified using a vector (recall our discussion of positions as vectors in Supplement 3). Commonly $\vec{r}$ is used (or $\mathbf{r}$; we'll use the boldface notation in the lecture slides because it's easier to do in PowerPoint!). If we have specified a coordinate system including the origin of the system, then in three dimensions we can represent $\vec{r}$ in terms of three components; for example, if we use a Cartesian system, then we could talk about the (x, y, z) position. Of course, the position is the position; it doesn't *require* that we define coordinates.

If we want to be careful we actually have to specify one more thing: the *time* at which we make the measurement. After all, the object could be moving, and if it is, then its position will change with time. Thus we could say that the position at a time t is $\vec{r}(t)$.

But the position by itself is not sufficient to describe the motion. The position *as a function of time* for all times of interest is sufficient, but it is more convenient to define quantities to indicate the rate of change of the position (i.e., the *velocity*), and to indicate the rate of change of the velocity (i.e., the *acceleration*).

This is where calculus comes in. Without calculus, you would have to ask "over what interval of time do I measure the change?" For example, should it be a second, a year, a millennium, or what? Then the rate of change would depend on the interval. For instance, if your particle is going around in a circle once per minute, and your interval is one minute, you would conclude that its position doesn't change at all. That's absurd.

Instead, as you recall, calculus asks the question: "what is the *ratio* of the change in the quantity of interest, to the time interval, in the limit that the time interval shrinks to zero?". For example, if we suppose that the position of the object at time t is $\vec{r}(t)$, and that its position at some later time $t + \Delta t$ is $\vec{r}(t + \Delta t)$, then we are interested in

$$\frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{(t + \Delta t) - t} \tag{1}$$

in the limit $\Delta t \to 0$. Although we expect that for real objects both $\vec{r}(t+\Delta t) - \vec{r}(t)$ and $(t+\Delta t) - t$

will approach zero as $\Delta t \to 0$, their *ratio* will approach a well-defined value for $\Delta t \to 0$. This value is what we call the velocity, and the velocity is also a function of time:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} , \tag{2}$$

where $d\vec{r}(t)$ is the value of $\vec{r}(t + \Delta t) - \vec{r}(t)$ in the limit $\Delta t \to 0$, and dt is the value of $(t + \Delta t) - t$ in the limit $\Delta t \to 0$. The notation $d\vec{r}(t)/dt$ is somewhat unfortunate, because we might have an urge to cancel the ds, but the d before a quantity indicates that we have taken the limit of small changes.

In physics and astronomy, the time derivative of a quantity is often denoted by putting a dot over the quantity. For example,

$$\dot{\vec{r}}(t) \equiv d\vec{r}(t)/dt , \tag{3}$$

where as you recall "≡" means "is defined as".

Similarly, the *acceleration* is the time derivative of the velocity:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\frac{d\vec{r}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \ddot{\vec{r}}(t) , \tag{4}$$

where the two dots at the end mean two time derivatives. We could likewise, define the time derivative of the acceleration, and the time derivative of that quantity, and so on, but people tend not to because force is proportional to acceleration and enough is enough (for the curious, the time derivative of the acceleration is usually called the jerk, and the successive time derivatives are semi-informally called the snap, crackle, and pop; clearly, some people have too much time on their hands!).

Another very important quantity related to the motion of an object is the *momentum*. In the Newtonian limit, where objects move slowly (more precisely: where objects move at much less than the speed of light), we can define the momentum as

$$\vec{p} = m\vec{v} , \tag{5}$$

where m, the mass of the object, is constant for such slow motion. However, the momentum is more general than that. For example, a photon, which moves at the speed of light all the time and has no "rest mass" (i.e., mass when it is stationary), has a momentum.

2. Newton's Laws of Motion

Newton's three laws of motion are often phrased in the following way:

1. An object moves at constant velocity unless a net force acts to change its speed or direction.

2. Force equals mass times acceleration.

3. For every force, there is an equal and opposite reaction force.

Let's express these using vectors and calculus; in the following expressions we won't continue to indicate explicitly the time dependence (so, for example, $\vec{v}(t) \rightarrow \vec{v}$), but we will understand that the time dependence is there anyway.

First, we note that Newton's first and second laws can be combined. The first law can be stated as "an object has constant momentum unless a net force acts on it". Let's hold that thought in mind. The second law, expressed in vectors, is

$$\vec{F} = m\vec{a} \; . \tag{6}$$

I claim that in the Newtonian limit for which the mass m is a constant, this is equivalent to

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{7}$$

where $\vec{p}$ is the momentum. To see this, we remember that $\vec{p} = m\vec{v}$ when we are in the Newtonian slow-motion limit (recall that this means that all speeds are much less than the speed of light). Thus

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \; . \tag{8}$$

The product rule for derivatives tells us that this is equal to

$$\frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v} \; . \tag{9}$$

But the mass m is constant, so $dm/dt = 0$. Thus the second term goes away. Also, we recognize that $d\vec{v}/dt = \vec{a}$, so we get finally

$$\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a} \; , \tag{10}$$

where the last equality holds for objects moving slowly compared with the speed of light. The form $\vec{F} = d\vec{p}/dt$ is actually more general than $\vec{F} = m\vec{a}$, and when we look at Newton's first law we see that it follows from his second law: if $\vec{F} = 0$, then $d\vec{p}/dt = 0$, which means that $\vec{p}$ is constant.

What about Newton's third law? Suppose that we have two objects, A and B. Let the force of A on B be $\vec{F}_{AB}$, and the force of B on A be $\vec{F}_{BA}$. Then Newton's third law says that

$$\vec{F}_{AB} = -\vec{F}_{BA} \; . \tag{11}$$

Multiplying a vector by -1 just flips its direction around without changing the length; thus the forces are of equal magnitude, but opposite direction. As one example of this that is not intuitively obvious, your gravitational force on the Earth has exactly the same magnitude (and the opposite direction) as the Earth's gravitational force on you! People not trained in physics would consider this ridiculous; it seems obvious that Earth's force on you would be greater than your force on the Earth. But no, the forces are equal in magnitude.

This is an amazingly general law! It tells us that for *any force whatsoever*, the equal-but-opposite law applies. This turns out to have important consequences for which quantities are constant in a given system, which we'll explore in the next supplement.

Angular momentum

You may remember that in Supplement 3 we defined the angular momentum $\vec{L}$ as

$$\vec{L} = \vec{r} \times \vec{p} \, , \tag{12}$$

where $\times$ is the cross product. We can therefore determine what would change the angular momentum. We do this by taking its time derivative:

$$\frac{d\vec{L}}{dt} = \frac{d(\vec{r} \times \vec{p})}{dt} \, . \tag{13}$$

The product rule for derivatives also works for cross products:

$$\frac{d(\vec{r} \times \vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \, . \tag{14}$$

Let's consider the first term. $d\vec{r}/dt = \vec{v}$, and we know that $\vec{p}$ is parallel to $\vec{v}$; for example, in the Newtonian limit, $\vec{p} = m\vec{v}$. But you may remember from Supplement 3 that the cross product of two parallel (or antiparallel) vectors is zero. Thus the first term vanishes. For the second, we can substitute $\vec{F}$ for $d\vec{p}/dt$, by Newton's second law. Therefore,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} \, . \tag{15}$$

The time derivative of the angular momentum is called the *torque*. From this formula, we can tell that if $\vec{F}$ is parallel or antiparallel to $\vec{r}$, then because the cross product is zero, the angular momentum is constant (that's because only the derivative of a constant is zero). Gravity acts along the line between two objects, which means that (by the definition of $\vec{r}$), the force of gravity is parallel or antiparallel to $\vec{r}$. As a result, for gravity, $\vec{r} \times \vec{F} = 0$, which is why the angular momentum of the planets in their orbit around the Sun is constant (caveat: here we neglect the other planets and assume that the Sun is much more massive than the planets).

Once more, please feel free to talk with the tutors, the TAs, or me about any aspect of this material. As part of that, you might enjoy coming up with thought experiments or challenges; contemplation of such matters is a great way to boost your intuition!

Practice problems

1. The three-dimensional position of a particle as a function of time t is $\vec{r}(t) = (3, 2t, 4t)$. Calculate the velocity of the particle.

Answer: velocity is the derivative of the position with respect to time. The derivative of a vector is the derivative of its components one by one. In this case, $r_x = 3$, which is a constant, so $v_x = dr_x/dt = 0$. $r_y = 2t$, so $v_y = dr_y/dt = 2$. $r_z = 4t$, so $v_z = dr_z/dt = 4$. Therefore $\vec{v} = (0, 2, 4)$.

2. The three-dimensional position of a particle as a function of time t is $\vec{r}(t) = (4t^3, 1 + t, 3t^2)$. Calculate the velocity and acceleration of the particle.

Answer: for the velocity, $v_x = dr_x/dt = d(4t^3)/dt = 12t^2$. $v_y = dr_y/dt = d(1 + t)/dt = 0 + 1 = 1$. $v_z = dr_z/dt = d(3t^2)/dt = 6t$. The acceleration is the time derivative of the velocity, so $a_x = dv_x/dt = d(12t^2)/dt = 24t$. $a_y = dv_y/dt = d(1)/dt = 0$. $a_z = dv_z/dt = d(6t)/dt = 6$. Thus $\vec{v} = (12t^2, 1, 6t)$ and $\vec{a} = (24t, 0, 6)$.

3. The three-dimensional position of a particle as a function of time t is $\vec{r}(t) = (t^4, 7t^2 + 2t, 2t^3)$. Calculate the velocity and acceleration of the particle.

4. The mass of a particle is $m = 2$ kg and the velocity is $\vec{v} = (2, 0, 1)$ m s^{-1}. *With units included*, compute the momentum $\vec{p}$ of the particle.

Answer: $\vec{p} = m\vec{v}$, so the momentum is $\vec{p} = (2\text{ kg})(2, 0, 1)$ m s^{-1} $= (4, 0, 2)$ kg m s^{-1}.

5. The mass of a particle is $m = 17$ kg and the velocity is $\vec{v} = (3, -1, 4)$ m s^{-1}. *With units included*, compute the momentum $\vec{p}$ of the particle.

6. A particle of mass $m = 3$ kg has a force $\vec{F} = (6, 0, 9)$ Newtons applied to it, where a Newton is the SI unit of force: 1 Newton=1 kg m s^{-2}. Calculate the acceleration $\vec{a}$ of the particle due to this force, *with units included.*

Answer: Newton's second law tells us that $\vec{F} = m\vec{a}$, so $\vec{a} = \vec{F}/m$. In our case, $\vec{a} = (6, 0, 9)$ kg m s$^{-2}/(3\text{ kg}) = (2, 0, 3)$ m s^{-2}.

7. A particle has a position $\vec{r} = (1, 2, 0)$ m and a momentum $\vec{p} = (3, 2, 0)$ kg m s^{-1}, as measured with respect to a particular point. Calculate the angular momentum $\vec{L}$ of the particle relative to that point.

Answer: $\vec{L} = \vec{r} \times \vec{p}$. $(1, 2, 0) \times (3, 2, 0) = 2\hat{z} - 6\hat{z} = -4\hat{z} = (0, 0, -4) = (0, 0, -4)$ kg m^2 s^{-1}.

8. A particle has a position $\vec{r} = (2, 3, 4)$ m and a momentum $\vec{p} = (1, 2, 3)$ kg m s^{-1}, as measured with respect to a particular point. Calculate the angular momentum $\vec{L}$ of the particle relative to that point.

9. Suppose that we have an object of mass m moving in the $x - y$ plane. Thus $z = 0$ for this object.

As a function of time, the position of the object is given by

$$\begin{aligned} x &= \cos \omega t \\ y &= \sin \omega t \ . \end{aligned} \tag{16}$$

Derive the velocity, acceleration, and momentum as a function of time. Also derive the *magnitude* of the velocity (i.e., the speed) as a function of time, as well as the magnitude of the acceleration. You should find that the speed is constant. The derivative of a constant is zero, so is the acceleration zero?