Flux and Seasons

1. Basics of flux

In these supplemental notes, we will discuss a concept that has often proved challenging: flux. In general, the flux of a quantity is the rate of the quantity per area, or equivalently **the amount of that quantity per time per area**. We will focus on energy flux:

- The *energy flux* at a surface is the energy per time per area that the surface receives. Thus in standard SI units, the units of energy flux are J s^{-1} m^{-2}.

The flux matters in many astronomical settings. For example, all else being equal, the more flux a surface receives, the more it radiates, and the hotter it is. This is the fundamental driver of seasons.

The definition is simple enough, but the difficulties come when we think about how to compute flux, because there are some geometrical subtleties. We'll begin with the easier part: *all else being equal, when you are farther from a source of energy the flux is less.* Thus distance spreads out flux.

We can get a sense of this by looking at Figure 1, which is from Wikipedia. The rays move in straight lines, and thus as they spread out from the source S the angle between them remains the same. Let us use the small angle approximation from Supplement 3, for which the angle subtended by something of projected size d at a distance $r \gg d$ is $\theta \approx d/r$. Then we can invert the expression: at a distance r from a source, an angular separation θ corresponds to a physical separation of $d \approx \theta r$. Thus the *area* spanning the rays has to be proportional to r^2 (as in the figure, think of this as a square; the angle along one side corresponds to a distance proportional to r, as does the angle along the other side, which means that the area is proportional to r^2). This is closely related to the area of a sphere of radius r, which as you know is proportional to r^2 (in fact, it's $4\pi r^2$).

If the energy per time is something (call it the luminosity L), then at larger radii the energy per time is spread over a larger area. In fact, the energy per time per area is just the energy per time *divided* by the area, which means that if the energy per time is the same at all radii (which is a good approximation in many cases) then the energy per time per area must scale like r^{-2}.

This is therefore the inverse square law for flux, which refines our earlier statement: *all else being equal, the flux is proportional to the inverse square of the distance from a source.*

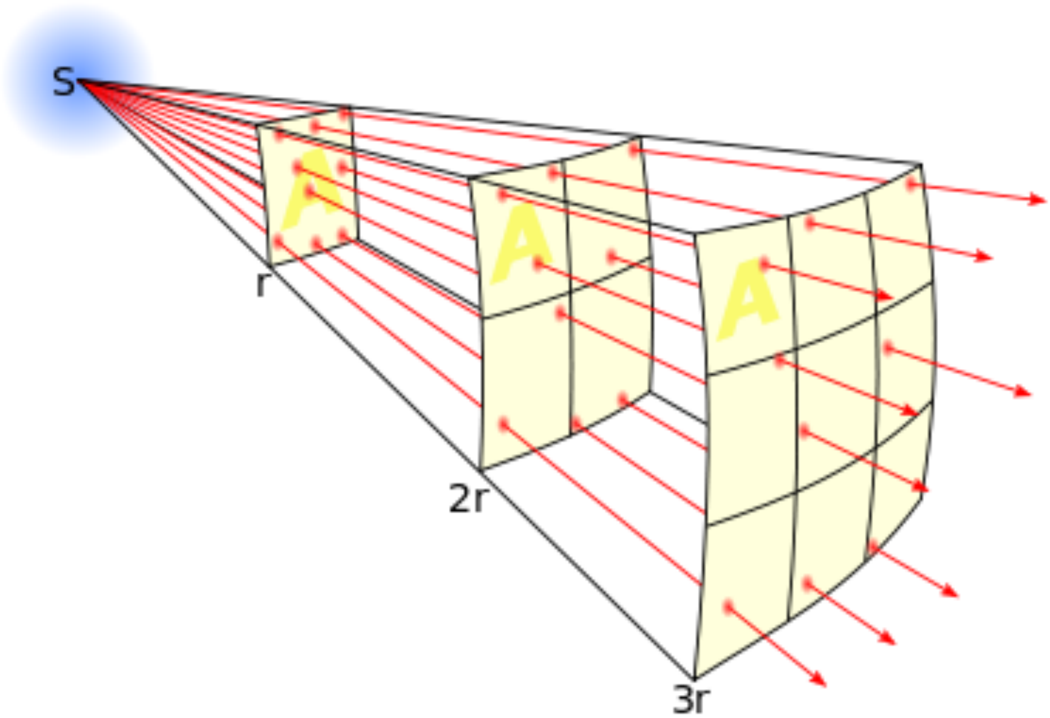

Fig. 1.— Representation of inverse square law, from Wikipedia. The source of light, which we think of as a point, is S. The different rays of light (there are nine here) move out from the source. The physical area that encompasses the rays increases with the square of the distance from the source (see the text for more of a justification of this). But it's the same amount of light, i.e., the same energy per time, so the flux (energy per time per area) has to decrease like the reciprocal of the square of the distance. Original figure from https://upload.wikimedia.org/wikipedia/commons/thumb/2/28/Inverse_square_law.svg/420px-Inverse_square_law.svg.png.

But there is another element, which is not as obvious: with flux, just as with the small angle formula we saw in the previous supplement, orientation matters. If we are interested in the flux received by a given surface from a point source, then the flux through *that surface* depends on the orientation of the surface. It is often very helpful for our insight to consider limits of a situation, so let's do that here. Suppose our surface is a flat square of some area. Call that area 1 m^2 if we want to be definite. If the square faces directly toward the point source, then 1 m^2 of light will be intercepted. But now let's tilt the surface. If we were to tilt the square so that it is edge-on to the point source, then the square would intercept no light at all. Thus the energy per time received by the square would be zero. But the square would still have the same area as before (1 m^2 in our example). Thus the flux would be zero. Orientation matters.

An equivalent way to think about this is suggested by Figure 2. If we think about a *given* bundle of rays, we can ask how large an area would be needed to intercept them all, at a specified tilt angle θ away from face-on. As we see in the figure, the more we tilt away from face-on, the more the area will need to be extended in the direction of the tilt to intercept the same rays.

Thus here we have a situation where, by assumption, the whole bundle of rays is intercepted by the surface. As a result, the energy per time intercepted by the surface is fixed. But the *area* of the surface is proportional to $1/\cos\theta$. **Therefore the energy per time *per area* is proportional to the energy per time (a constant) divided by $1/\cos\theta$, and therefore the flux is proportional to $\cos\theta$.**

A concept that is useful to introduce at this stage is the *normal* vector to a flat surface, sometimes just called the "normal". It is an outward-pointing vector that is perpendicular to the surface. For example, if the surface is in the $x-y$ plane, then the normal points along the z axis.

Let's see how this applies to a sphere because, after all, the Earth is close to spherical and we'd ultimately like to figure out the main cause of our seasons. Consider Figure 3. We'll imagine for the moment that the Earth's rotation axis aligns with its orbital axis; in the next section we will consider the actual case, which is that the Earth's rotation axis is tilted by 23.5° to its orbital axis.

Building on on the material in the previous supplement, let us define the z axis to point from the center of the Earth through the Earth's north pole, and define the x axis to point toward the Sun. Recall from the last supplement that it can be useful to define coordinates on a sphere such that θ (the colatitude) is the angle between a point and the north pole (so that θ runs from 0 to π radians), and ϕ (the longitude) is 0 in the $x-z$ plane and runs from 0 to 2π radians. Thus the unit vector toward the Sun is $\hat{v}_{\rm Sun} = (1,0,0)$ (because in our definition the Sun is in the direction of the x-axis), and the unit vector to a point at colatitude θ in the $x-z$ plane toward the Sun is $\hat{v}_{\rm surface} = (\sin\theta, 0, \cos\theta)$ (remember that the y-component is $\sin\theta\sin\phi$, so for $\phi = 0$ that component vanishes).

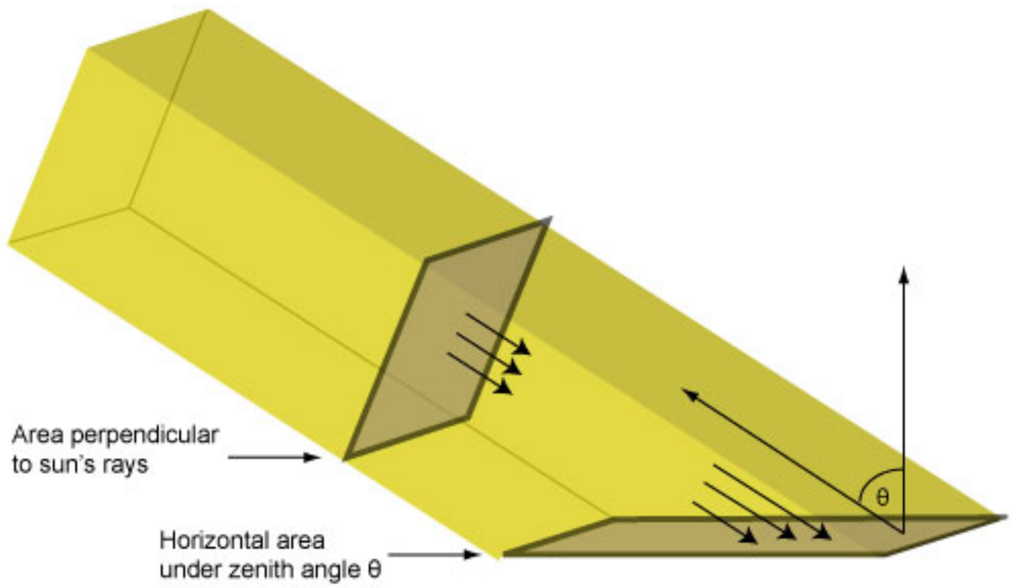

Fig. 2.— Demonstration that more inclined surfaces will receive smaller flux at a fixed distance from a point source of radiation (represented by the Sun in this case). We consider a set of parallel light rays. All of them are intercepted both by the face-on square on the left, and the inclined rectangle on the right. But in order to intercept all the rays, the inclined rectangle must be extended (by a factor of $1/\cos\theta$). Thus although both the square and the rectangle intercept the same energy per time, the flux (i.e., the energy per time per area) at the inclined rectangle is smaller simply because its area is larger. Original figure from http://www.greenrhinoenergy.com/solar/radiation/images/flux-01.jpg.

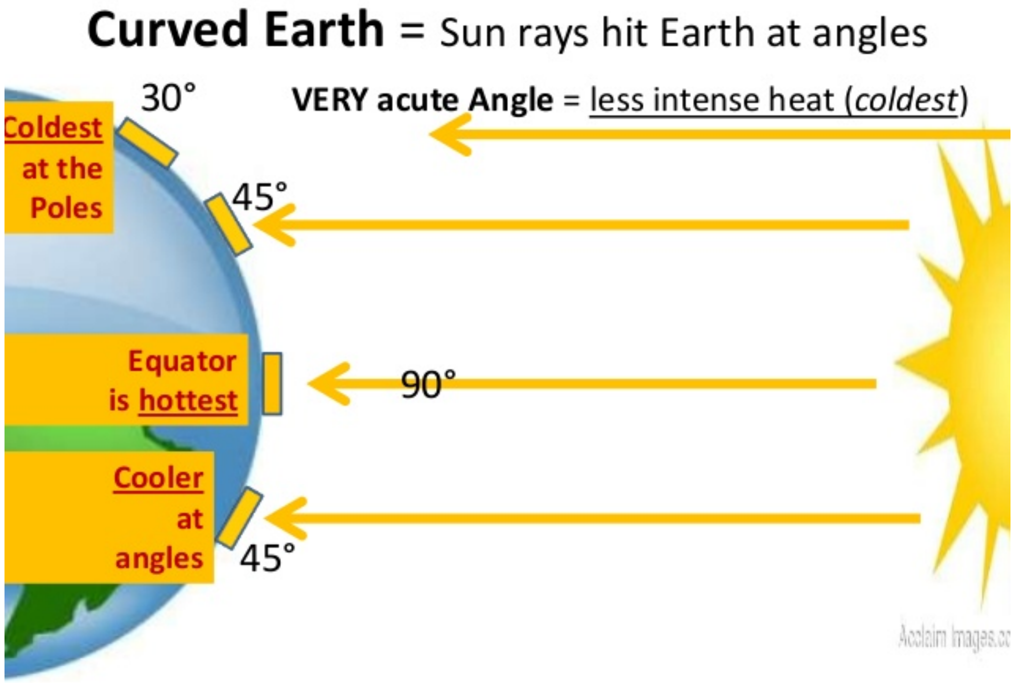

Fig. 3.— Diagram showing that less direct rays lead to less heating, which is why the poles are colder than the equator on Earth. Original figure from https://image.slidesharecdn.com/solarenergyfinal-131104011343-phpapp02/95/solar-energy-and-its-affect-on-earths-atmosphere-19-638.jpg?cb=1383527857

Consider for example a location on the Earth with a latitude of 40° north, which is close to where we are in College Park. This being the latitude means that's the angular distance between us and the equator. But the *colatitude* is measured from the north pole, so because the north pole has a latitude of 90° north, this location would have a colatitude of $90^\circ - 40^\circ = 50^\circ$. If the location had had a latitude of 40° *south*, then its colatitude would have been $90^\circ - (-40^\circ) = 90^\circ + 40^\circ = 130^\circ$. At the moment when our location faces as directly as possible toward the Sun, then its longitude would be 0°. Twelve hours later, the location would have rotated exactly halfway around, so that its longitude would be 180°, or π radians. We'll focus on the situation in which locations are facing as directly as possible toward the Sun given their colatitude. At that time, therefore, the unit vector from the center of the Earth to our location is $\hat{v}_{\rm surface} = (\sin 50^\circ, 0, \cos 50^\circ) \approx (0.766, 0, 0.643)$.

From our dot product formula in section 2.2 of the previous supplement, we know that the dot product between two unit vectors is the cosine of the angle between the vectors (let's call that $\cos\psi$). But this is exactly what we want to know for the flux, which as we found above is proportional to the cosine of the angle between the normal to the surface and the direction of the rays! Thus we get

$$\cos\psi = (1, 0, 0) \cdot (\sin\theta, 0, \cos\theta) = \sin\theta\ . \tag{1}$$

Let's see if this makes sense. We know that the poles should be colder than the equator, which means that the flux at the poles should be smaller than the flux at the equator. Is that what this formula says? The formula says that at $\theta = 0$ or $\theta = \pi$ (respectively, the north and south poles), $\cos\psi = 0$, whereas at $\theta = \pi/2$ (the equator), $\cos\psi = 1$. Thus the formula does indeed predict that the poles are colder than the equator.

To summarize this section, the flux received by a surface depends on both the distance to the source (as well, obviously, as the luminosity of the source) and the angle between the normal to the surface and the direction to the source.

2. Seasons

Now we can think about seasons. Many people, when asked the reason for seasons on Earth, will say that we are closer to the Sun in the summer, and farther in the winter. But we know this can't be the whole story. If that were the only factor, then countries in the southern hemisphere (such as Australia) would have the same seasons that we do in the northern hemisphere. In fact, they have the opposite seasons, so something else is going on.

That other factor is the angle of the rays to the surface. This angle changes with the seasons, and in this section we'll approach this quantitatively.

Let the tilt angle of the Earth's rotation relative to its orbital axis be $\theta_{\rm tilt}$; in fact, $\theta_{\rm tilt} = 23.5^\circ$. Let the colatitude of the observer be $\theta_{\rm obs}$; for those of us in College Park, Maryland, where the *latitude* is 39° north, the *colatitude* is $90^\circ - 39^\circ = 51^\circ$.

When the orbit of the Earth is such that the north pole points as closely as it can to the direction of the Sun, then at noon the normal vector from College Park is pointed as closely as possible to the direction of the Sun. At that time, relative to a z-axis which we define as perpendicular to the x-axis that points to the Sun, the colatitude is $\theta = \theta_{\rm obs} + \theta_{\rm tilt} = 74.5^\circ$. Six months later, when the north pole points as far away as possible from the direction of the Sun, at noon the angle is instead $\theta = \theta_{\rm obs} - \theta_{\rm tilt} = 27.5^\circ$. This demonstrates something that you might not have realized: that in College Park, the Sun is *never* directly overhead, which would be $\theta = 90^\circ$! Many people, if asked, might guess that the Sun is directly overhead at noon every day, or that it is directly overhead at noon on the summer solstice, but it's not true in College Park.

This tells us that the effect of *tilt* at the latitude of College Park is to change the flux by a ratio of $\sin(74.5°)/\sin(27.5°) \approx 2.1$.

What about the effect of *distance*? Over its orbit, the Earth's distance from the Sun varies by a factor of 0.967 (that's the ratio of its smallest distance to its greatest distance). The flux depends on the inverse square of the distance, which means that when we are farthest from the Sun, the distance effect alone produces a flux ratio of $1/0.967^2$, or 1.069. This is very close to 1; clearly, at our latitude, the tilt angle effect is *way* more important than the changing distance!

For further thought: As you know, close enough to the north pole or south pole there are parts of the year that the sun never rises. And yet, those parts of the Earth do not fall in temperature to absolute zero! For that matter, fortunately, we don't drop to absolute zero at night :). What are some of the factors that keep the surface somewhat warm at night? As you consider your ideas, what would you predict about the relative differences between day and night temperatures on the Moon, compared with the Earth? What about Venus? What properties of these bodies might be important?

As before, please don't hesitate to talk with the tutors, your TAs, or me about what you find interesting about this topic, or about specific topics for which you would like an additional or different explanation.

Practice problems

1. Saturn's average distance from the Sun is 9.54 times Earth's average distance from the Sun. Calculate the ratio of the solar flux received by a surface face-on to the Sun at Saturn's distance, to the solar flux received by a surface face-on to the Sun at Earth's distance.

Answer: For a fixed angle of the surface to the light source, the flux depends on the inverse square of the distance. Therefore the flux at Saturn is $1/9.54^2 \approx 1/91$ of the flux at Earth.

2. Do the same calculation as above for the ratio of the flux at Jupiter to the flux at Earth, again assuming in both cases that the surface is face-on to the Sun. Jupiter's average distance from the Sun is 5.20 times Earth's average distance.

3. Saturn's radius is 9.45 times Earth's radius (we're treating Saturn as a sphere; in reality is is a bit oblate). Calculate the ratio of the solar *luminosity* intercepted by Saturn to that of Earth.

Answer: we found before that the flux at Saturn is about 1/91 of the flux at Earth. But Saturn's area is proportional to its radius squared, so its area is $9.45^2 \approx 89.3$ times Earth's area. The luminosity is the flux times the area, so the luminosity intercepted by Saturn is about $89.3 \times (1/91.0) \approx 0.98$ times the luminosity intercepted by Earth.

4. Do the same calculation as above for the ratio of the luminosity intercepted by Jupiter to that of Earth. Jupiter's radius is 11.21 times that of Earth.

5. Convince yourself, using geometry, that for a surface tilted an angle θ away from face-on, the area needed to intercept a given bundle of rays must be proportional to $A \propto 1/\cos\theta$. Does this expression work in limits such as $\theta \to 0$ (face-on) and $\theta \to \pi/2$ (edge-on)?

6. You orient a detector face-on to the Sun and find that the detector intercepts a flux F_0. At the same time and location, your friend orients a detector at 60° from face-on. What flux does your friend measure?

Answer: for a fixed distance, the flux is proportional to the cosine of the angle away from face on. $\cos(60°) = 1/2$, so your friend measures a flux $F_0/2$.

7. The total area of your friend's detector is three times as large as your detector. Compare the *luminosity* that your friend's detector intercepts to the luminosity yours intercepts.

Answer: again, the luminosity is the flux times the area. If we say that your detector has 1 unit of area, then it intercepts $F_0 \times 1$ units of luminosity. Your friend's detector intercepts $(F_0/2) \times 3 = (3/2)F_0$ units of luminosity, so your friend's detector intercepts 3/2 times as much luminosity as yours does. Note that (3/2) in the above expression has units of area, e.g., square meters; flux and luminosity do *not* have the same units.

8. This time, your detector is oriented at 60° from face-on, and your friend's detector is oriented at 30° from face-on. Also, this time your friend's detector has half the area that yours does. Calculate

the ratio of fluxes, and the ratio of luminosities, seen by your detector and your friend's.

9. We know that a sphere of radius R has an area $4\pi R^2$, and thus that half a sphere has an area of $2\pi R^2$. But what is its *projected* area, i.e., the area it would appear to have if you looked at it from a very large distance? Here we are asking for the physical area; that is, if you are at a very large distance $r \gg R$, and you see that the angular distance as you see it from the center of the sphere to its edge is θ, then the physical area you would infer is $\pi(r\theta)^2$. Does your answer make physical sense?

10. Miami is at a latitude of 25.8° north. Calculate the ratio of fluxes due to the angle, as we did for College Park. Anchorage is at a latitude of 61.2°. Calculate the ratio of fluxes due to the angle for Anchorage. What conclusion can you draw about how the flux ratio depends on the latitude, i.e., is it a larger ratio for places close to the equator or close to the poles? What if the place in question is *on* the rotational equator (Quito, Ecuador is pretty close)?

11. If you want to stretch yourself, note that we talk above about the situation at local noon. But the Earth rotates, and at most latitudes, there will come a time when the Sun appears to disappear below the horizon. We call this phenomenon "night" :). Can you calculate the duration of night as a function of latitude, for the two special times in the year we discussed above (north pole pointed maximally toward Sun; north pole pointed maximally away from the Sun)? **Hint:** if you want to do this problem, I suggest that you (a) define the z-axis to to be pointing toward the Earth's north pole at all times), (b) define the Sun to be in the $x - z$ plane, and thus ϕ for the Sun is zero, but then θ for the Sun would be $\pi/2 - \theta_{\rm tilt}$ when the north pole points toward the Sun, and $\pi/2 + \theta_{\rm tilt}$ when the north pole points away. In addition, if we define $\phi = 0$ as local noon, then as the Earth rotates, $\phi = t/(24 \text{ hours})$, where t is the time since noon. Using this, you should be able to write equations that you could in principle solve to get a dot product of zero between the normal and the direction to the Sun; the solutions will define dawn and dusk.