Geometry and Vectors

In these supplemental notes, we will explore some astronomically relevant issues related to geometry and vectors. Geometry is important to astronomy mainly because angles between things on the sky are so useful. Vectors, which are quantities that have both magnitude and direction, are critical to physics in innumerable contexts.

1. Geometry: angular size, angular differences, and the small-angle approximation

Some objects in the sky are so far away relative to their sizes that even with telescopes we can't see any details in their images: they look like points. Most stars are like that. In contrast, there are plenty of objects that *can* be resolved with good enough telescopes. For example, galaxies fall into that category, although the smallest and most distant galaxies might require large telescopes. Other astronomical objects, such as the Moon, are close enough relative to their size that even with our eyes we can see structure.

Note, though, that whether we can see structure depends not on the *absolute* size of an object, but on its *angular* size: its size relative to its distance. Consider Figure 1 as a reference: we are at point a, and we are looking at some object that is at a distance D from us and spans the distance between point c and point d.

The opening angle near vertex a is the angular size of the whole object. Half that angle is therefore obviously half the angular size of the whole object.

Digging back into our geometry, we remember that if abc is a right angle, then the half-angle (which we can call θ) is the angle such that $\tan\theta = \frac{cb}{ab}$, where cb and ab are the lengths of the sides; in our case, we imagine that the distance ab is equal to D.

But this is the geometry we have if the object is oriented perpendicular to our line of sight. There are other possibilities. For example, look at Figure 2. This is actually taken from a website that shows how to do a certain geometric construction, which is why there are extra lines, but we'll use it for our purposes. Here we see that if we are at point C and the distance CO is equal to D, then the angle θ that we see made by side OM is such that $\sin\theta = OM/D$. That is, it's a sine rather than a tangent.

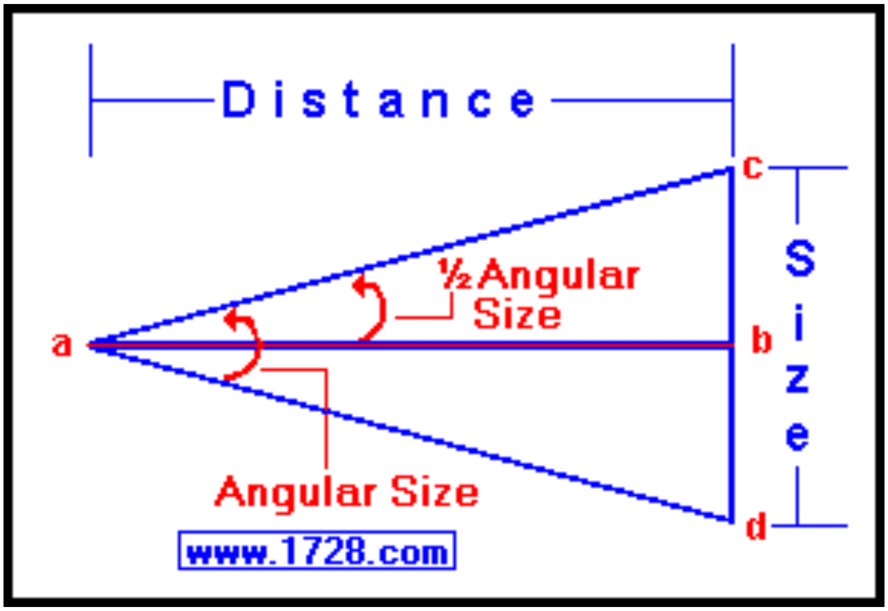

Fig. 1.— Diagram of angular size. The angle θ subtended by cb as seen from a is given by $\tan\theta = \frac{cb}{ab}$, where cb and ab are the lengths of the sides. Original figure from http://www.1728.org/angsize.png.

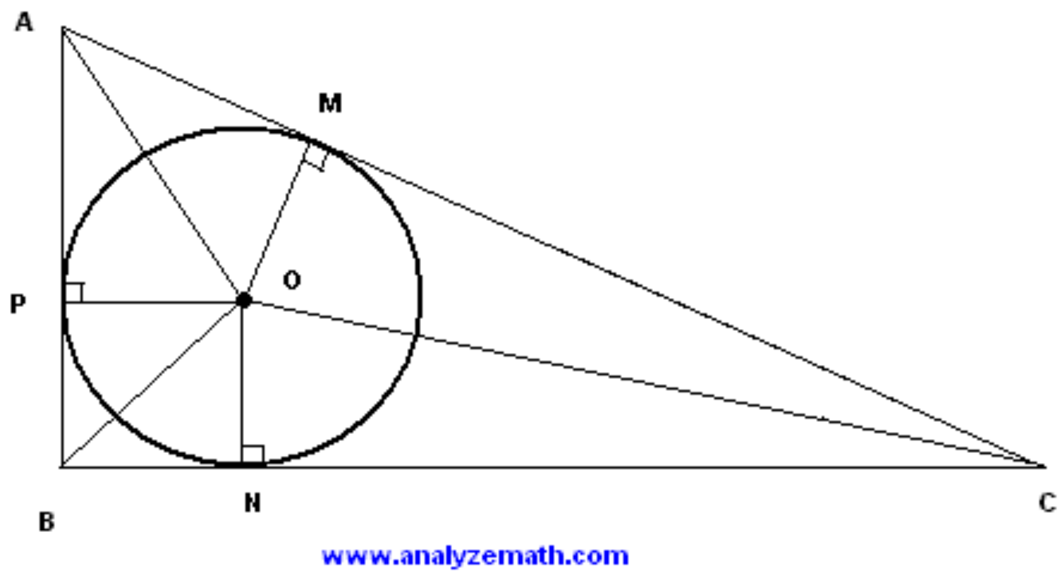

Fig. 2.— Another instance of angular size, but this time requiring a sine: the angle θ subtended by the side OM as seen from point C is given by $\sin\theta = \frac{OM}{CO}$, where again OM and CO are the lengths of the sides. Original figure from http://cdn-5.analyzemath.com/Geometry/circle_within_right_tri_sol.gif.

For very close things, this can make a difference. For example, suppose we want to determine the largest angle that Venus can appear to make away from the Sun, as seen from Earth. If we assume that the orbits of both Venus and Earth are circles, and if we remember that **the tangent to a point on the circumference of a circle is always perpendicular to the radius that touches that point** (as we see in Figure 2; for the Earth-Venus-Sun example, Earth would be at C, Venus would be at M, and the Sun would be at O), then we would need a sine. If the object is flat and perpendicular to us, we would need a tangent. Typically in astronomy we measure distances between the centers of objects, e.g., between the center of the Earth and the center of the Sun.

Stated like that it might appear that to determine the angular size of something (which we might do to figure out whether we can resolve it in an image) we need to consider special cases involving different types of trigonometry.

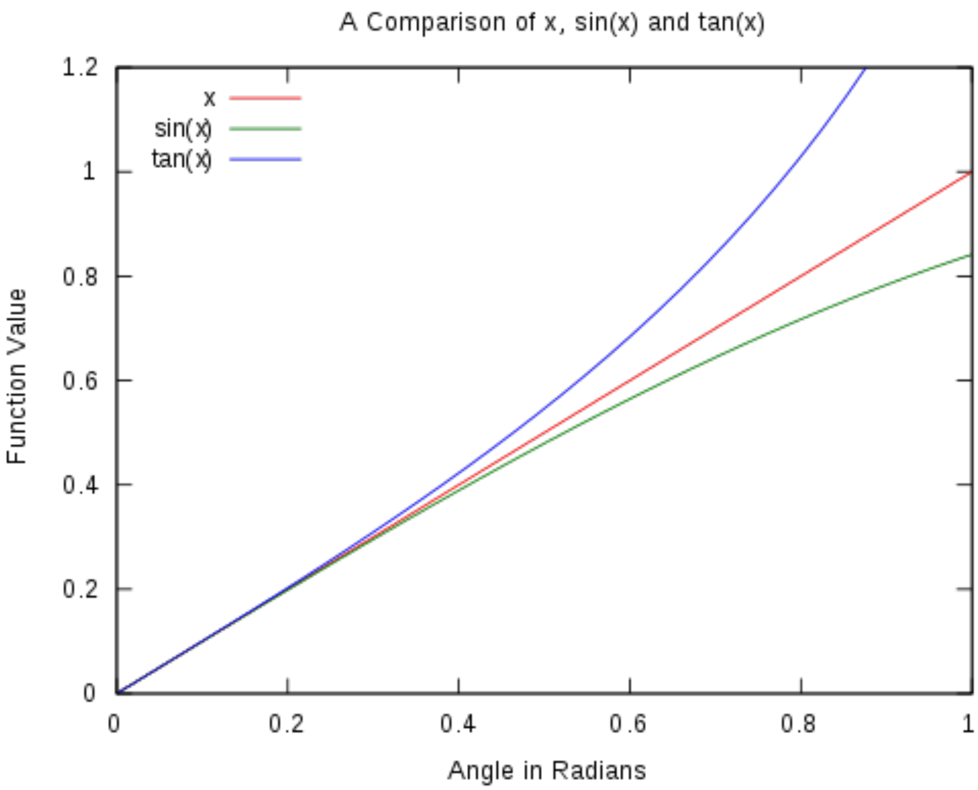

Fig. 3.— Motivation for the small-angle approximation $x \approx \sin x \approx \tan x$ when the angle x (in radians) is much less than 1. This figure, from Wikipedia, shows the three quantities as a function of x. Original reference: https://upload.wikimedia.org/wikipedia/commons/thumb/1/16/Small_angle_compair_odd.svg/500px-Small_angle_compair_odd.svg.png.

Luckily, however, **most angular sizes in astronomy are small**. As a result, it is usual that people apply the *small angle approximation*. Suppose that we measure θ in radians. Then you might remember that for $\theta \ll 1$, $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$. Thus if an object has size s perpendicular to your line of sight and is a distance D from you, then the angle it subtends is

$$\theta \approx s/D \ . \tag{1}$$

Figure 3 shows that for a small argument x (x is θ for our case), x, $\sin x$, and $\tan x$ are all very close to each other. If you have taken calculus, you might remember that to the next order of detail, $\sin x = x - x^3/6 + \ldots$, and $\tan x = x + x^3/3 + \ldots$ Thus if $\theta = x$ is even moderately small, the differences between θ, $\sin\theta$, and $\tan\theta$ are tiny. For example, if $\theta = 0.1$ (more than ten times the angular size of the Moon!), then $\sin\theta = 0.09983\ldots$ and $\tan\theta = 0.1003\ldots$

The "perpendicular to your line of sight" bit is important; if you have a narrow rod of length s that is at a distance D from you, its angular size is $\theta \approx s/D$ if it is oriented perpendicular to your line of sight, but if the rod is pointed toward you then its angular size is much smaller.

You can usually use the small-angle approximation in astronomy, but as with any formula, please check to see that it applies before using it! For any approximation, "see that it applies" is a *quantitative*, not a *qualitative* criterion. What I mean by that is that no approximation is perfect, and you'll need to judge whether for your particular purpose the error you make by using the approximation is acceptable. For example, we saw above that for $\theta = 0.1$, $\sin\theta = 0.09983\ldots$ and thus the relative error is $(0.1 - 0.09983)/0.09983 = 0.00167\ldots \approx 0.17\%$. Thus if an error of less than 0.2% doesn't matter to you, you can use the small-angle approximation when $\theta < 0.1$. If there is some purpose for which you need better than 0.01% precision, then you would have a stricter threshold on θ.

Our final point along these lines is that in addition to often wanting to know the angular *size* of something, we even more often want to know the angular *distance* between two things. The definition follows naturally, but it is good to remember that we can define an angular distance between two things at very different distances from us. This is, for example, relevant for constellations. We see patterns of stars in the sky, and those stars seem close together in an angular sense, but it is common that despite their angular closeness, two stars in a constellation are at vastly different distances. For example, Bellatrix, which is one of the shoulder stars in Orion, is at about 200 light years distance. Betelgeuse, which is the other shoulder star, is at 640 light years distance. But the angular distance between them (sometimes called their angular *separation*) is small.

2. Vectors

2.1. Basics of vectors

A vector is something that has both magnitude and direction. Vectors are used all the time in physics. For example, a velocity requires a magnitude (which is the speed) as well as a direction.

Similarly, momentum, force, acceleration, and many other quantities are described using vectors.

Three numbers are required to describe a vector in three dimensions (if you are working in two dimensions, e.g., in a plane, then only two numbers are needed. In general, in n dimensions you need n numbers). What those three numbers are depends on the coordinate system we are using. For example, if we are interested in a velocity in a Cartesian xyz system, we could indicate the speed along x, the speed along y, and the speed along z. Or, we could indicate the total speed, and then use the other two numbers to indicate the direction (think about it: something like a latitude and a longitude will tell you the direction uniquely). A *unit vector* is a vector whose magnitude is 1; this will come in handy later in this supplement when we talk about spherical geometry.

When we write vectors using Cartesian components, the notation for a vector $\vec{v}$ is often $\vec{v} = (v_x, v_y, v_z)$, where v_x is the x-component of the vector, v_y is the y-component, and v_z is the z-component (note that sometimes vectors are indicated by boldface, i.e., $\mathbf{v}$ rather than $\vec{v}$ — unfortunately there isn't a universal notation — but we'll use an arrow to make it a bit clearer). For example, if you have defined the x-direction as due east, the y-direction as due north, and the z-direction as straight up, a velocity

$$\vec{v}_2 = (1, 2, 3)\ \mathrm{m\ s^{-1}} \tag{2}$$

would mean a speed of 1 m s^{-1} due east, a speed of 2 m s^{-1} due north, and a speed of 3 m s^{-1} straight up, all at the same time. A velocity

$$\vec{v}_2 = (-1, 2, 3)\ \mathrm{m\ s^{-1}} \tag{3}$$

would then mean a speed of 1 m s^{-1} due *west* (because west is opposite to east), a speed of 2 m s^{-1} due north, and a speed of 3 m s^{-1} straight up, all at the same time. If you have defined different coordinate directions (e.g., x could be north and y could be west) then a given set of numbers will mean a different thing, but in three dimensions you need three numbers.

The *magnitude* of a vector is its total length. For this, if you have expressed the vector in terms of components, you just use the Pythagorean Theorem in three dimensions:

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}\ . \tag{4}$$

The vertical bars around the vector indicate that we mean the magnitude of the vector. It is also common to just omit the bars and the vector sign: $v = |\vec{v}|$. Note that the length of a vector is independent of the coordinates we use. For example, speed is a vector; it is the magnitude of velocity. If I ask Usain Bolt to run as fast as he can, then his speed will not depend on how I set up my x, y, z coordinate system!

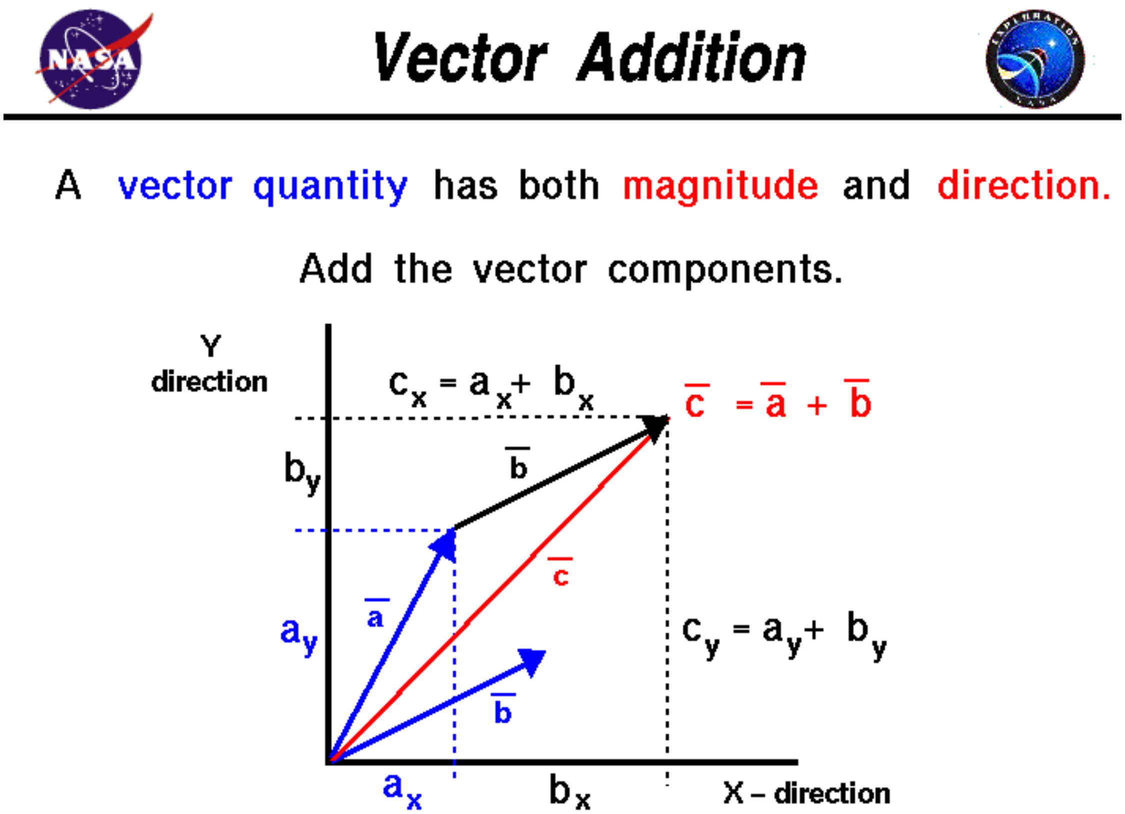

Fig. 4.— Addition of two vectors, using components, in two dimensions. Note that once you have defined your x and y axes, you need to add the x-components of the vectors together, and separately add the y-components of the vectors together. Original reference: https://www.grc.nasa.gov/WWW/K-12/rocket/Images/vectadd.gif.

In order to familiarize you with different notation, we note that a different way to write using components involves *unit vectors*, which are vectors with length 1 in your measurement system. These are often written using "hats" over the vectors, e.g., $\hat{x}$ is a unit vector in the positive x direction, so that $\hat{x} = (1, 0, 0)$. Similarly, $\hat{y} = (0, 1, 0)$ and $\hat{z} = (0, 0, 1)$. Thus a general vector can be written as

$$\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \ . \tag{5}$$

Sometimes in books, $\hat{\imath}$ will be used instead of $\hat{x}$, $\hat{\jmath}$ instead of $\hat{y}$, and $\hat{k}$ instead of $\hat{z}$. Sorry about that :)

If you have written a vector in Cartesian components, then adding a second vector to the first requires that you add the components of the two vectors. An example in two dimensions is given in Figure 4. Generally, if you use components, then in three dimensions the sum of two vectors $\vec{A}$ and $\vec{B}$, which have components in some Cartesian system $\vec{A} = (A_x, A_y, A_z)$ and $\vec{B} = (B_x, B_y, B_z)$, is given by

$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z) \ . \tag{6}$$

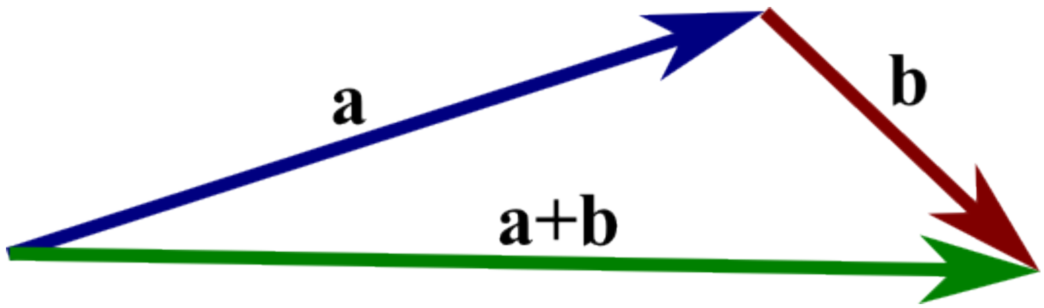

Fig. 5.— A different look at adding vectors. Rather than defining a coordinate system and adding components, we can think of just putting the tail of one vector at the head of another. Original reference: https://mathinsight.org/media/image/image/vector_a_plus_b.png.

But you actually don't need to define a coordinate system. As is also shown in Figure 4, and as is shown more simply in Figure 5, the more geometrical way to add vectors is to put the tail of one at the head of the other. Either order works; if $\vec{A}$ is one vector and $\vec{B}$ is another, then $\vec{A}+\vec{B} = \vec{B}+\vec{A}$. The independence of ordering can also be seen from the components: for example, you know that $A_x + B_x = B_x + A_x$ if A_x and B_x are any normal numbers.

As an example of the addition of vectors, let's suppose that I am running with a velocity $\vec{v}_1$ with respect to you, and I throw a ball that has velocity $\vec{v}_2$ as I see it. If we are working in the Newtonian limit (where both speeds are tiny compared with the speed of light), then you see the velocity of the ball to be

$$\vec{v}_{\rm ball} = \vec{v}_1 + \vec{v}_2 \ . \tag{7}$$

2.2. Dot products of vectors

There are two additional important ways to combine vectors: *dot products* and *cross products*. Here we treat dot products, and in the next subsection we treat cross products.

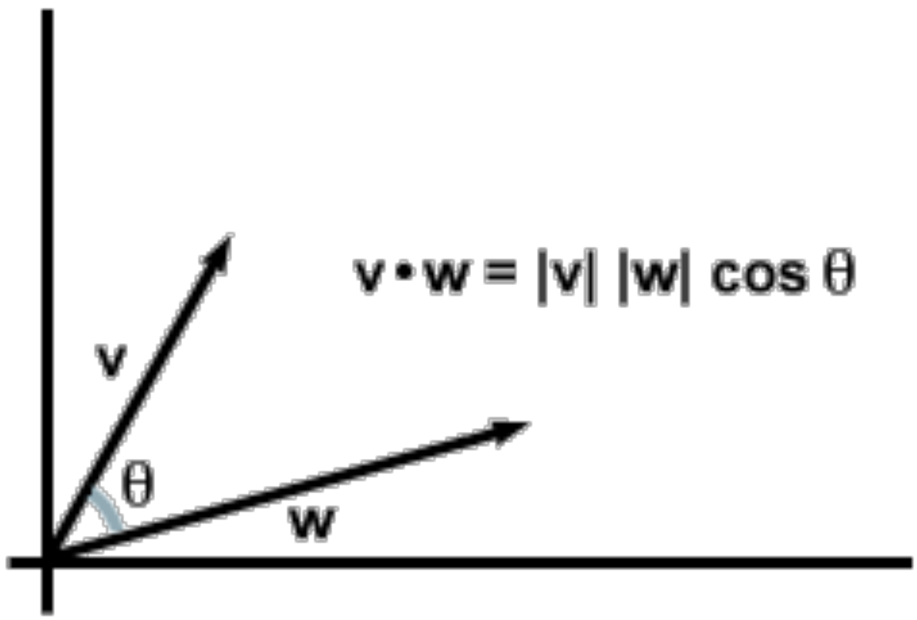

Fig. 6.— Dot product between two vectors. The result of a dot product is a pure number rather than being a vector: its magnitude is the product of the magnitudes of the two vectors, times the cosine of the angle between them. The dot product $\vec{v}\cdot\vec{w}$ is equal to the dot product $\vec{w}\cdot\vec{v}$. Original reference: http://img.sparknotes.com/figures/1/13493b46f82b15be90229290a86eb26a/dotproduct.gif.

To understand dot products, look at Figure 6. We have vectors $\vec{v}$ and $\vec{w}$. The dot product $\vec{v}\cdot\vec{w}$ is a *scalar*, not a vector; **a scalar has magnitude but *not* direction**. For example, mass is a scalar: you might have a mass of some number of kilograms, but there isn't a direction associated with it.

In this figure, $|v|$ means the magnitude of $\vec{v}$, and $|w|$ means the magnitude of $\vec{w}$. Then $\vec{v}\cdot\vec{w} = |v||w|\cos\psi$ if ψ is the angle between $\vec{v}$ and $\vec{w}$. You can see, therefore, that if two vectors are parallel to each other ($\psi = 0$; their tail to head directions are identical), then their dot product is just the product of their magnitudes. If two vectors are perpendicular to each other ($\psi = \pi/2$), then their dot product is zero. If two vectors are *antiparallel* to each other ($\psi = \pi$; the tail to head direction of one is the head to tail direction of the other), then their dot product is the negative of the product of their magnitudes. You can also see that $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

If we use components, then the dot product of two vectors is the sum of the products of their individual components. Thus if $\vec{v} = (v_x, v_y, v_z)$ and $\vec{w} = (w_x, w_y, w_z)$, then

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z \ . \tag{8}$$

With this definition, you can see that the dot products of our coordinate unit vectors is zero, because they are perpendicular to each other. For example:

$$\hat{x} \cdot \hat{z} = (1,0,0) \cdot (0,0,1) = 1 \times 0 + 0 \times 0 + 0 \times 1 = 0 \ . \tag{9}$$

One possibly familiar example of a dot product has to do with the rate at which work is done on something by a force. Suppose that the force is $\vec{F}$ and the object has a velocity $\vec{v}$. Then the rate of work (i.e., energy per time put into the object) is $\vec{F} \cdot \vec{v}$. If the force is in the direction of motion, then $\vec{F}$ is parallel to $\vec{v}$ (or $\vec{F} \parallel \vec{v}$) and the object gains energy. If the force is opposite the direction of motion, then $\vec{F}$ is antiparallel to $\vec{v}$, and the object loses energy. Something that might not be obvious, but that is nonetheless true, is that if $\vec{F}$ is perpendicular to $\vec{v}$, the object neither gains nor loses energy no matter how large the magnitude of $\vec{F}$ is!

Challenge: use this concept to solve the following problem. Draw an arbitrarily convoluted track on a piece of paper: you can put in loops, sharp turns, whatever you like. Suppose that a car moves along this track, and its speed is the same everywhere on the track. Given this information, draw the direction of the force on the car at several random places.

An important but inobvious use of dot products is that if you know the magnitudes of the vectors and the value of their dot product, then from the dot product formula you can determine the angle between the two original vectors. That's because

$$\begin{aligned} \vec{v} \cdot \vec{w} &= |v||w| \cos\psi \\ \cos\psi &= (\vec{v} \cdot \vec{w})/(|v||w|) \ . \end{aligned} \tag{10}$$

This will come up again when we explore spherical geometry later in this supplement.

2.3. Cross products of vectors

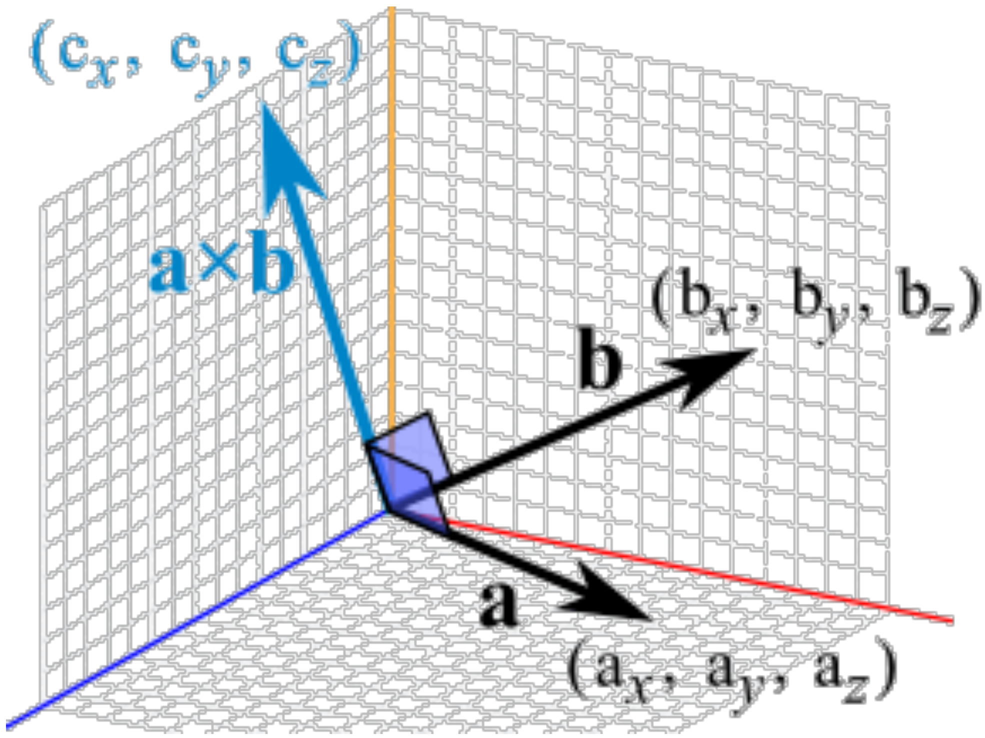

Fig. 7.— Cross product between two vectors. The result of a cross product is another vector, which is perpendicular to both of the original vectors. The magnitude of the cross product is the product of the magnitudes of the two vectors times the sine of the angle between them. Thus the cross product of vectors that are parallel, or antiparallel, is zero. The convention is to use the "right hand rule": if you put the back of your right hand along the direction of $\vec{a}$, and curl your fingers in the direction of $\vec{b}$, then your right thumb points in the direction of the cross product. As a result, if you reverse the order you get a minus sign: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Original reference: https://www.mathsisfun.com/algebra/images/cross-product-components.gif.

To get a sense for cross products, look at Figure 7. **The result of a cross product between two vectors is another vector, which is perpendicular to each of the two original vectors. The magnitude of the vector is the product of the magnitudes of the two vectors, times the sine of the angle between them.** The direction of the cross product vector is given (by convention) by the right hand rule; see the caption to Figure 7. This means that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.

Writing a cross product using components is a little more challenging, but it is important to know how to do. If you are comfortable with matrices, and in particular the calculation of their determinants, then that's an easy way to get cross products; see, for example, http://tutorial.math.lamar.edu/Classes/CalcII/CrossProduct.aspx.

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x, A_y, A_z) \times (B_x, B_y, B_z) \\ &= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\ &= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}\ .\end{aligned} \tag{11}$$

There are a lot of terms there. It may be helpful to think about this in terms of unit vectors. For a right-handed system xyz (which by convention is almost always used), xy is positive and yx is negative; yz is positive and zy is negative; and zx is positive and xz is negative. For example, let's perform the cross product between $\vec{A} = (1, 0, 0) = \hat{x}$, and $\vec{B} = (0, 1, 0) = \hat{y}$. When we go through formula 11 we find that

$$\vec{A} \times \vec{B} = \hat{x} \times \hat{y} = (0, 0, 1) = \hat{z}\ . \tag{12}$$

Notice that, as promised, the result ($\hat{z}$) is perpendicular to both of the vectors used in the cross product ($\hat{x}$ and $\hat{y}$). Also as promised, the magnitude is the product of the magnitudes of the original vectors (both 1, since these are unit vectors) and the sine of the angle between them (the angle between $\hat{x}$ and $\hat{y}$ is $90°$, and $\sin 90° = 1$): the magnitude of $\hat{z}$ is 1, as required.

When we do the product in the opposite order, we find

$$\vec{B} \times \vec{A} = \hat{y} \times \hat{x} = (0, 0, -1) = -\hat{z}\ . \tag{13}$$

I encourage you to do such calculations with other pairs of coordinate unit vectors, for example, $\hat{x} \times \hat{z}$, and then to try some more complicated vectors. You might also try cross products of coordinate unit vectors with themselves, e.g., $\hat{y} \times \hat{y} = 0$.

An example of the cross product in physics is the angular momentum, which is often represented by $\vec{L}$. If you define a reference point, and relative to that reference point an object has a location $\vec{r}$ and a linear momentum $\vec{p}$, then relative to the reference point the object has an angular momentum

$$\vec{L} = \vec{r} \times \vec{p}\ . \tag{14}$$

For example, the Earth has an orbital angular momentum around the Sun. If it moves in a perfect circle, then its linear momentum $\vec{p}$ is always perpendicular to its radius vector from the Sun $\vec{r}$, and then the magnitude of its angular momentum is just $|\vec{r}||\vec{p}|$. For something going directly away from the Sun, $\vec{p}$ is parallel to $\vec{r}$, and that object would have zero angular momentum. You can also see this using components; if, for example, we define the x direction such that $\vec{r} = |r|\hat{x}$, and the momentum is parallel to this, so that $\vec{p} = |p|\hat{x}$, then $\vec{r} \times \vec{p} = |r||p|\hat{x} \times \hat{x} = 0$. The angular momentum is a very important concept; for isolated systems, the total angular momentum of the system does not change. We'll revisit this later in the course.

3. Spherical geometry

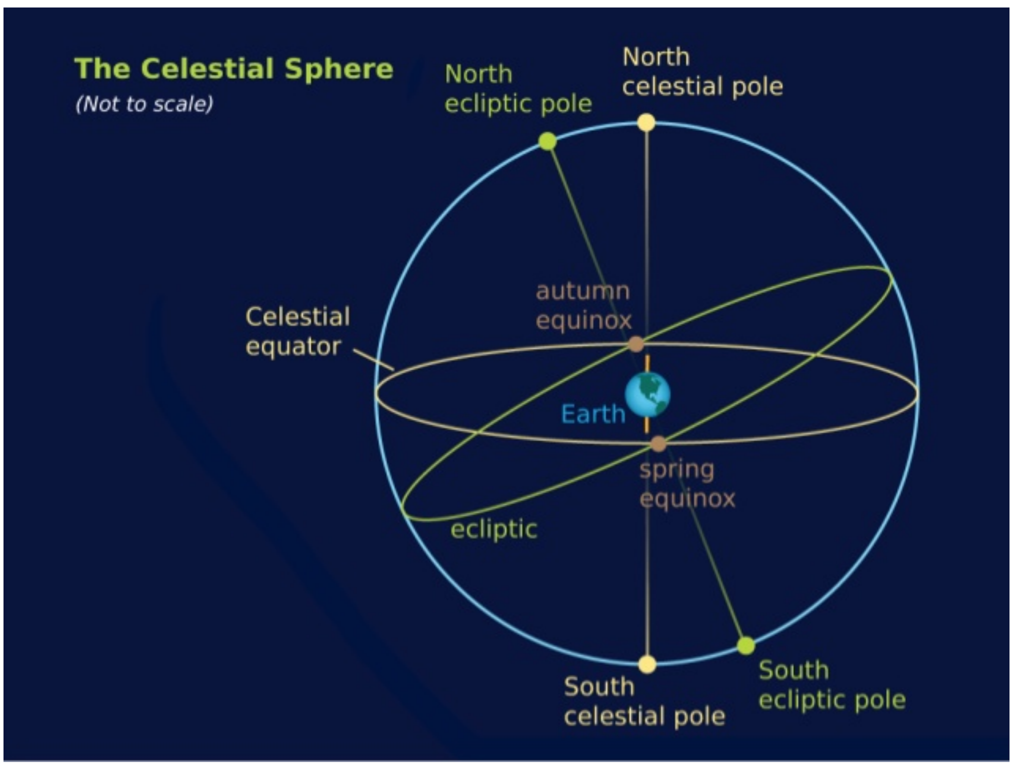

Fig. 8.— Basics of the celestial sphere. When we specify directions we use a system of longitude and latitude (called right ascension and declination in this case) where we imagine ourselves to be at the center of a great sphere. The north celestial pole is the extension of a line from the center of the Earth through our north rotational pole, and similarly with the south celestial pole. Original reference: https://image.slidesharecdn.com/celestialsphere-160305053122/95/the-celestial-sphere-1-638.jpg?cb=1457160426

Since ancient times it has been useful to imagine that we on Earth are at the center of a great sphere: the celestial sphere. Part of our discussion in class will be focused on how this is defined, but you can get a preview in Figure 8: the north rotational pole of the Earth points toward the "North celestial pole", and likewise the south rotation pole of the Earth points toward the "South celestial pole". How fortunate we are to be at the origin of the system for the whole cosmos! :) In any case, we use this system to identify the directions to things in the sky.

One of the most observationally useful calculations is the angular distance between two directions on the sky. For example, when NASA plans observations of different objects using a space telescope, they need to take angular distance into consideration because it is much better to move the direction of a telescope by small amounts than by large amounts. Therefore, observatory planning needs to incorporate information about the angular distance between successive targets.

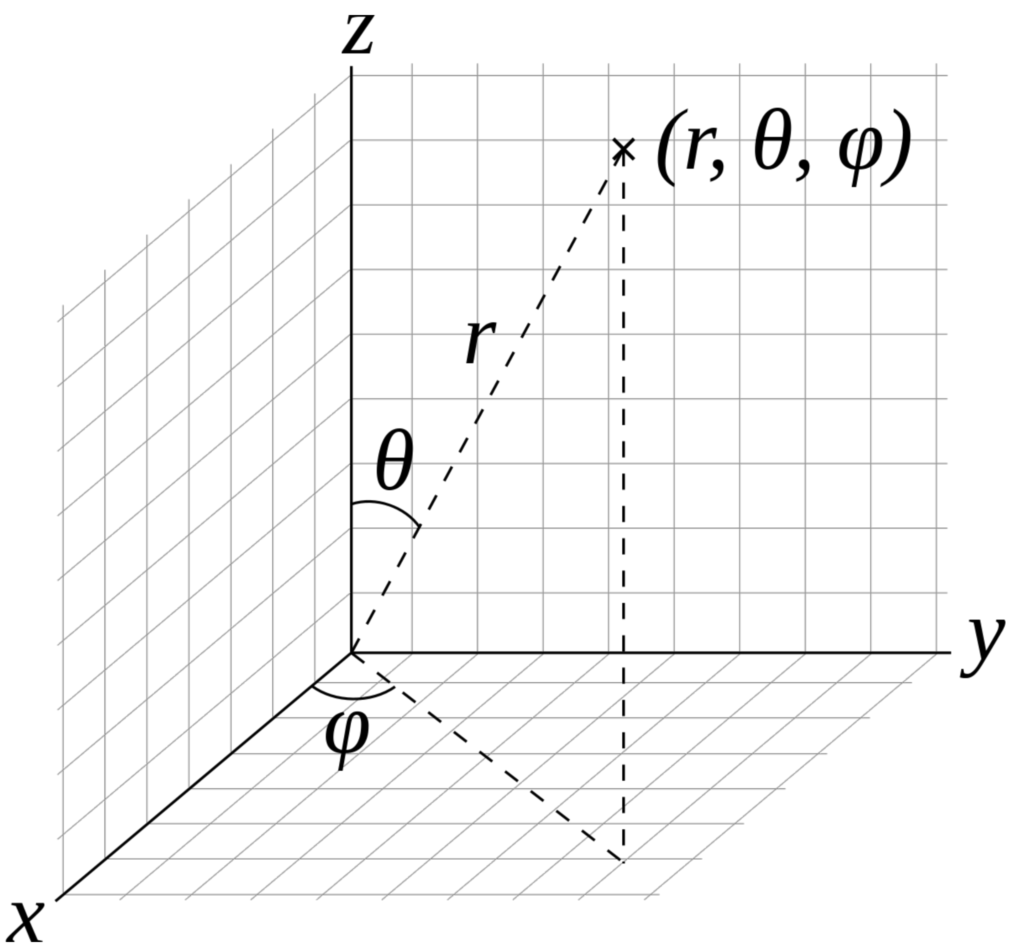

Fig. 9.— Spherical coordinates, from Wikipedia. Here r is the distance from the origin, θ is the angle from the positive z axis, and ϕ is the angle, from the projection in the xy plane, counterclockwise from the x axis. Note that the latitude is measured from the xy plane, but θ is measured from the north pole; thus θ is called the colatitude. Original reference: https://upload.wikimedia.org/wikipedia/commons/thumb/4/4f/3D_Spherical.svg/1200px-3D_Spherical.svg.png

But when we think about doing this for a sphere, it might not be clear how to calculate angular distances. As a start, we need to think about how to define directions on a sphere. To do this, we reiterate that we imagine that we are at the center of the sphere. Then we can define the direction to something by its latitude and longitude. If necessary, we can also specify the distance to the object. Thus we arrive at the definition of *spherical coordinates*, which are represented in Figure 9. Here r is the distance from the origin and ϕ plays the role of longitude. ϕ can run from 0 to 2π radians. Latitude is measured from the equator; positive latitudes for the northern hemisphere and negative latitudes for the southern hemisphere. In physics applications it is usually more convenient to use the definition indicated in the figure, where θ is measured from the positive z axis rather than the equator; θ is called the *colatitude*, and it runs from $\theta = 0$ (positive z axis) to $\theta = \pi/2$ (equator) to $\theta = \pi$ (negative z axis).

A unit vector has length 1 by definition, so $r = 1$. We assert that the unit vector in direction (θ, ϕ) has components

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ . \tag{15}$$

Now we are set up to determine the angle between any two directions on the sky, which we can label (θ_1, ϕ_1) and (θ_2, ϕ_2). Remember our dot products? We gave two expressions for the dot product between any two vectors $\vec{v}$ and $\vec{w}$:

$$|\vec{v}||\vec{w}|\cos\psi \ , \tag{16}$$

where ψ is the angle between the vectors, and

$$v_x w_x + v_y w_y + v_z w_z \ , \tag{17}$$

where (v_x, v_y, v_z) and (w_x, w_y, w_z) are the components of each vector. But here we are considering unit vectors $\hat{v}$ and $\hat{w}$, and by definition $|\hat{v}| = |\hat{w}| = 1$. Thus the angle ψ between our unit vectors is given by

$$\cos\psi = v_x w_x + v_y w_y + v_z w_z \ , \tag{18}$$

where the components of the unit vectors $\hat{v}$ (in direction (θ_1, ϕ_1)) and $\hat{w}$ (in direction (θ_2, ϕ_2) are given by equation 15.

I know we've gone through a lot here. I hope that at least some of this is review rather than new, but regardless of the situation I strongly urge you to learn this material. It is important for observational astronomy (particularly the bits about spherical geometry), and for theoretical astrophysics (where vectors are used all the time). Also note that there are many resources you can tap, both on the Web (e.g., Khan Academy and Wikipedia) and at UMd (our tutors, TAs, and me). I wish you the best, and please don't hesitate to ask questions!

Practice problems

In these supplements I will give you a number of practice problems, which I recommend that you solve because doing problems will give you a deeper and longer-lasting understanding of the subjects than simply reading about them. Some of the practice problems will involve simple application of the concepts in that supplement, whereas others might ask you to think a little more deeply. Because many of you will not have encountered vectors, and because it is a mathematical subject that is important for many subjects in the rest of the class, we'll have a lot more practice problems here than in other supplements.

1. Suppose that a narrow rod, which has a length s and is a distance D from you, is oriented at an angle ϕ relative to your line of sight. What is its angular size then, if $s \ll D$?

2. For the following cases, let $\theta = s/D$ and compute $\sin\theta$ and $\tan\theta$ to determine how much they differ from θ itself. (a) $s = 1.08 \times 10^8$ km and $D = 1.496 \times 10^8$ km (the orbit of Venus, as seen from the Earth), (b) $s = 1,737$ km and $D = 384,000$ km (the radius of the Moon and the distance of the Moon from the Earth), (c) $s = 8.2 \times 10^8$ km and $D = 6.08 \times 10^{15}$ km (radius of Betelgeuse and its distance from us).

3. For the next five problems, let vector $\vec{A} = \hat{x} - 2\hat{y}$ and vector $\vec{B} = \hat{x} + 2\hat{y}$, where $\hat{x}$ is the unit vector along the x direction and $\hat{y}$ is the unit vector along the y direction, and as usual the x and y directions are perpendicular to each other. For this problem, calculate the magnitude of $\vec{A}$ and the magnitude of $\vec{B}$.

Answer: The magnitude of a vector is its total length, or $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ in three dimensions. For $\vec{A}$, this is $\sqrt{1^2 + (-2)^2} = \sqrt{5}$. For $\vec{B}$, this is $\sqrt{1^2 + 2^2} = \sqrt{5}$. These two vectors happen to have the same magnitude.

4. Calculate $\vec{A} + \vec{B}$.

Answer: $\vec{A} + \vec{B} = (\hat{x} - 2\hat{y}) + (\hat{x} + 2\hat{y}) = \hat{x} + \hat{x} - 2\hat{y} + 2\hat{y} = 2\hat{x}$.

5. Calculate $\vec{A} - \vec{B}$.

Answer: $\vec{A} - \vec{B} = (\hat{x} - 2\hat{y}) - (\hat{x} + 2\hat{y}) = \hat{x} - \hat{x} - 2\hat{y} - 2\hat{y} = -4\hat{y}$.

6. Calculate $\vec{A} \cdot \vec{B}$.

Answer: Remember that the dot product of vectors $\vec{v} = (v_x, v_y, v_z)$ and $\vec{w} = (w_x, w_y, w_z)$ equals $v_x w_x + v_y w_y + v_z w_z$. Here we have no z components, so we just add the products of the x and y components: $\vec{A} \cdot \vec{B} = 1 \times 1 + (-2) \times 2 = 1 - 4 = -3$.

7. Calculate $\vec{A} \times \vec{B}$.

Answer: Recall that, generally, the cross product in terms of components is $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$. Here $A_z = B_z = 0$, which means that the cross product has no $\hat{x}$ term and no $\hat{y}$ term; if you think about it, this had to be the case because the cross product of two vectors is a vector that is perpendicular to both of the original vectors, and indeed the $\hat{z}$ direction is perpendicular to both $\hat{x}$ and $\hat{y}$. For the $\hat{z}$ component, we have $A_x B_y - A_y B_x = (1)(2) - (-2)(1) = 4$, so the final cross product is $4\hat{z}$.

8. Let $\vec{A} = \hat{x} + 2\hat{y} + 3\hat{z}$, and $\vec{B} = 2\hat{x} + 4\hat{y} + 6\hat{z}$. Compute $\vec{A} \times \vec{B}$, and explain the answer. By simply looking at $\vec{A}$ and $\vec{B}$ could you have found the answer without doing any calculating?

9. Remember that we also write vectors in the form (a, b, c), which would mean that the vector is $a\hat{x} + b\hat{y} + c\hat{z}$, where again $\hat{x}$, $\hat{y}$, and $\hat{z}$ are unit vectors along the coordinate directions, so that $\hat{x}$, $\hat{y}$, and $\hat{z}$ are all mutually perpendicular to each other. Write $(1, 2, 3)$ in $\hat{x}, \hat{y}, \hat{z}$ form. Write $2\hat{x} - 2\hat{y} + 5\hat{z}$ in (a, b, c) form.

Answers: $(1, 2, 3) = \hat{x} + 2\hat{y} + 3\hat{z}$. $2\hat{x} - 2\hat{y} + 5\hat{z} = (2, -2, 5)$.

10. Let $\vec{A} = (3, -1, 2)$ and $\vec{B} = (1, -2, -3)$. Calculate $|\vec{A}|$, $|\vec{B}|$, $\vec{A} + \vec{B}$, $\vec{A} - \vec{B}$, $\vec{A} \cdot \vec{B}$, and $\vec{A} \times \vec{B}$.

11. Can you prove, using components, that $\vec{V} \times \vec{V} = 0$ for *any* vector $\vec{V}$?

12. $\vec{A} = (1, 2, 0)$ and $\vec{B} = (1, a, 0)$. What is a so that $\vec{A} \cdot \vec{B} = 0$? What is a so that $\vec{A} \times \vec{B} = 0$?

13. $\vec{A} = (1, 1, 0)$ and $\vec{B} = (2, 0, 0)$. Calculate $\vec{A} \cdot \left(\vec{A} \times \vec{B}\right)$. Note that the order of operations is as in ordinary arithmetic, i.e., you need to compute the cross product first because it is in parentheses. After you get the result, then you compute the dot product.

14. $\vec{A}$ and $\vec{B}$ are general vectors, in three dimensions. What is the value of $\vec{A} \cdot \left(\vec{A} \times \vec{B}\right)$? **Hint:** think about how a cross product relates to the two original vectors.

15. $\vec{A} = (2, 0, 0)$, $\vec{B} = (0, 3, 0)$, and $\vec{C} = (0, 0, 1)$. Calculate $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$.

Answer:

$\vec{B} \times \vec{C} = (B_y C_z)\hat{x}$, because B_y is the only nonzero component of $\vec{B}$ and C_z is the only nonzero component of C. Thus $\vec{B} \times \vec{C} = 3\hat{x} = (3, 0, 0)$. Then $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = (2, 0, 0) \cdot (3, 0, 0) = 6$.

16. In the previous problem, suppose that we consider $\vec{A}$, $\vec{B}$, and $\vec{C}$ to be the three legs of a rectangular solid. What geometrical quantity associated with this solid is given by $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$? Test your hypothesis by computing $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ when $\vec{A} = (3, 0, 0)$, $\vec{B} = (0, 4, 0)$, and $\vec{C} = (0, 0, 2)$. What if $\vec{A} = (1, 0, 0)$, $\vec{B} = (2, 0, 0)$, and $\vec{C} = (0, 4, 0)$; does your geometrical quantity identification still make sense?

17. Suppose that $\vec{A}$, $\vec{B}$, and $\vec{C}$ are at general angles to each other. You can still form a solid (a parallelepiped) using the three vectors as legs. What, in general, is $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$ in that circumstance?

18. Remember that in spherical coordinates, a unit vector in direction (θ, ϕ) has components $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. From Figure 9 it should be that $\theta = 0$ corresponds to a unit vector

along the z direction. Can you verify that this is what the components say?

19. From the same figure, the xy plane must have $\theta = \pi/2$. If $\theta = \pi/2$, the figure also indicates that $\phi = 0$ corresponds to a unit vector along the x direction, and $\phi = \pi/2$ corresponds to a unit vector along the y direction. Can you verify that the components give these results?

20. Can you prove that for any θ and ϕ, the components of the spherical unit vector give a vector of length 1, i.e., a unit vector?