Using Units, Limits, and Symmetries

To kick off our supplementary notes, I will borrow extensively from a book that Doug Hamilton and I are preparing, about how to use units, limits, symmetries, and other quick checks to assess answers and to build physical intuition.

When solving physics problems it's easy to get overwhelmed by the complexity of some of the concepts and equations. It's important to have ways to navigate through these complexities and reduce errors. One of the best navigation tools is a sense of what the answer should look like. What units should it have? How should it behave in easily-understood limits? What are the symmetries of the problem? What should the answer depend on? You should check every answer you get against these common-sense guides. This will cut down dramatically on errors in derivation. Even more importantly, it will help build up your intuition about physics, because you will be able to approach problems by constraining the answer first. So let's get to it!

1. Checking Units

Units are the first thing to check when considering possible answers to a problem. Any equation that you write must be dimensionally correct. Check your equations occasionally as you go through a derivation. It takes just a second to do so, and you can quickly catch many common errors. Remember this general rule: in all physically valid solutions, the argument of trigonometric functions, exponentials, logs, hyperbolic functions, etc. must be dimensionless. Taking the cosine of something with units of mass or length makes no physical sense. In a similar way, if you are solving for the energy of an oscillation and come up with an answer that has units of momentum, that answer must be wrong.

Imagine that you and several friends have just gone through lengthy derivations and have come up with a number of different answers. Only one can be right, but how do you know which answers (and derivations) are suspect? Testing the units of the final answer can provide clues - if the dimensions of an answer do not match those of the quantity that you are looking for, then that answer (and the accompanying derivation) are wrong. Checking units will never tell you that an answer must be right, but it can tell you that an answer must be wrong. If a final answer is dimensionally incorrect, intermediate results in a derivation can also be tested; this can often help to rapidly pinpoint the source of the error.

Here's an example:

1. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange, d, does the daredevil fly before hitting the ground?

A) $d = 2v^2 \cos\theta$
B) $d = (2v^2/g)\sin\theta\cos\theta$
C) $d = 2g\sin\theta\cos\theta$
D) $d = 2vg(\cos\theta - \sin\theta)$
E) $d = (2v^2/g)\sin g$

Answer: Distance is measured in meters, velocities in m/s, and acceleration in m/s^2. All of the above answers have left-hand sides which are distances in units of meters, so the correct answer must have units of meters on the right as well.
A) has units of speed squared (WRONG)
B) has units of meters (COULD BE OK)
C) has units of acceleration (WRONG)
D) has units of (meter per second) squared (WRONG)
E) the argument of the sine has units (WRONG)
Note that units checking like this is important but does have limitations. For example, any equation that is dimensionally correct is also dimensionally correct if either side is multiplied by an arbitrary dimensionless factor.

Now it's your turn:

Challenge:
2. What is the maximum height h attained by the daredevil in problem 1?

A) $h = 2v^3 \sin\theta$
B) $h = (2g/v^2)\sin\theta$
C) $h = 2v\sin\theta\cos\theta$
D) $h = 2vg^2(\cos\theta - \sin\theta)$
E) $h = (v^2/g)\sin\theta$
F) $h = (2v^2/g^2)\sin g$

2. Checking Limits

Dimensional analysis of an expression is a skill that, once mastered, should always be used. It does have serious limitations, however, and cannot, for instance, tell you when you have dropped a factor of 2 or $\cos\theta$. More sophisticated tests are needed in these cases to catch additional errors and to bolster your confidence in answers that passes all tests. One of the most powerful ways to further test your answer is to consider key limiting cases. Always check your final answers and important intermediate results to see if they behave correctly in as many different limiting situations as you can imagine. Sometimes you will know how a general expression should behave if a particular variable is set to zero, infinity, or some other key value. Make sure that your general expression actually displays the expected behavior! You need to be very careful with limits; sometimes you may run into a paradox if you erroneously convince yourself that an expression must behave a certain way in a certain limit. It is always worthwhile to resolve these paradoxes, as this will help you build up your physical intuition. As with dimensional analysis, careful checking of limits can tell you that an expression is wrong, but it cannot prove that an expression is correct. In this section we focus on using limits wisely, but do not forget to also check units as you did in the previous section.

As our example, let's address our previous example problem using limit arguments.

3. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange d does the daredevil fly?

A) $d = 2v^2 \cos\theta$
B) $d = (2v^2/g)\sin\theta\cos\theta$
C) $d = 2g\sin\theta\cos\theta$
D) $d = 2vg(\cos\theta - \sin\theta)$
E) $d = 2v^2 \sin g$

Answer: First, consider the expected result in various limiting angles. If the cannon is pointed straight upward ($\theta = 90°$), no horizontal progress is made, so $d = 0$. An answer that predicts otherwise is wrong, so (D) is eliminated. Also, although (E) is meaningless because a dimensional quantity is the argument of a sine, it can also be eliminated by limits because it predicts no dependence on angle. When the cannon is pointed horizontally ($\theta = 0$), the daredevil will fall immediately so again $d = 0$. Answer (A) therefore is incorrect. Now let's think of what happens when the speed v is changed. When $v = 0$, we must have $d = 0$. Answer (C), though, has no v dependence, so it can be eliminated. We find, as we did by using units, that only B) can be correct.

For your problem, please check both units and limits.

Challenge:
4. Two bodies of masses m_1 and m_2 are placed a distance r apart. What is the strength F of the gravitational force that the bodies exert on each other?

A) $F = G(m_1 + m_2)m_2/r^2$
B) $F = G(m_1 + r)(m_2 + r)/r^2$
C) $F = Gr^2 m_1 m_2$
D) $F = G/(r^2 m_1 m_2)$
E) $F = Gm_1 m_2/r^2$

3. Taking Advantage of Symmetries

Symmetries are fundamental in physics (and astronomy!). Problems can have symmetry about a point (spherical symmetry), a line (cylindrical or axial symmetry), or a plane (mirror symmetry). You can use symmetries in two ways: 1) to check your final answer to a problem or, with a little more effort, 2) to simplify the derivation of that final answer. As an example, time-independent central forces (like gravity) have spherical symmetry because the force depends only on the distance from the origin. In this case, spherical symmetry means that once we find one solution (e.g. a particular ellipse for gravity), all other possible orientations of this solution in space are also solutions.

Another type of symmetry could be called a symmetry of labeling. In many problems, it is clear that simply renaming two identical things can't change anything fundamental about the system. For example, consider two objects of mass m_1 and m_2 moving in circular orbits around each other, bound by gravity, separated by a distance a. What is the frequency of rotation? A guess like $\omega = \sqrt{G(2m_1 + m_2)/a^3}$ can't be right, because the answer would change simply by switching the labels on the masses.

As an example:

5. A star of average radius R is rotating with angular frequency ω. We define the sign of ω such that if $\omega > 0$ then the star is rotating west to east like the Earth, whereas if $\omega < 0$ then the star is rotating east to west. Rotation will distort the radius of the star. To lowest order in ω, what will be the deviation ΔR of the equatorial radius from R?

A) $\Delta R \propto \omega$
B) $\Delta R \propto \omega^2$
C) $\Delta R \propto \omega^3$

Answer: B) is correct. Imagine looking at this star, then looking at it after standing on your head. It will appear to have switched directions of rotation ($\omega \to -\omega$), but the magnitude is unchanged. Clearly, Δr can't change because you stood on your head! That means that odd powers of ω are forbidden, so only B) is possible.

Now it's your turn!

Challenge:
6. You have coffee in a circular cup. The coffee is initially at rest, at height h_0. You stir it so that the coffee acquires a uniform angular velocity ω (here $\omega > 0$ if the motion is counterclockwise as seen from above, and $\omega < 0$ if the motion is clockwise). The coffee rises at the sides of the cup as a result, by an amount Δh. Which of the following could be true?

A) $\Delta h \propto \omega$
B) $\Delta h \propto \omega^2$
C) $\Delta h \propto \omega^3$

And finally, a grand challenge for you, which will require careful thought but which is a good test of your ability to apply high-level physical reasoning.

Challenge:

7. You launch a rocket straight up from the Earth's North pole and it rises up then falls back to Earth. The maximum height above Earth's surface h is given by one of the expressions below. Here R_E is the Earth's radius, $X = v^2 R_E / G M_E$, G is the gravitational constant, M_E is the Earth's mass and v is the launch speed. Rule out as many of the following expressions as you can.

A) $h = R_E X/(1 + \sqrt{X})$
B) $h = R_E X/(1 - X)$
C) $h = R_E X/(2 - X)$
D) $h = R_E(1 - X)/(2 - X)$
E) $h = vX^2/(2 - X)$
F) $h = R_E X/2$
G) $h = R_E X^2/(2 - X)$
H) $h = R_E X|1 - X|/(2 - X)$

As I will repeat for each of the supplemental notes, if there are aspects of a given set of notes that interest you particularly, or are unclear, or that stimulate further thought, please feel free to talk with the tutors, the TAs, or me!