ASTR 121
Quantitative Practice Problems

The idea of these problems is to give you practice in manipulating equations. I'll start out by giving you a few equations and tips, then go on to the problems.

There are some equations we have encountered enough times in ASTR 120 and ASTR 121 that they should now be familiar to you. Examples are:

Generalized Kepler's third law

$$P = 2\pi\sqrt{a^3/(GM_{\rm tot})} \ . \tag{1}$$

Here a is the semimajor axis, $M_{\rm tot}$ is the total mass of a binary, and P is the orbital period. Note that if you measure P in years, a in AU, and $M_{\rm tot}$ in solar masses, then you have

$$P({\rm yr}) = \sqrt{a^3({\rm AU})/M_{\rm tot}(M_\odot)} \ . \tag{2}$$

That is, written this way you don't have to worry about the 2π or Newton's constant G. **Hint:** when you have an equation like this, you can check it by finding out if it gives the expected answer in a known case. For example, you know that the Earth goes around the Sun in one year, so putting in $a = 1$ AU and $M_{\rm tot} = 1\,M_\odot$ had better give you one year!

Speed of circular orbit of radius r, of a particle of very small mass m around an object of much greater mass M.

$$v = \sqrt{GM/r} \ . \tag{3}$$

Conservation of momentum. If you have many objects in a system, the sum of their masses times their velocities is a constant. Momentum is a *vector* quantity, meaning that it has both magnitude and direction. If the mass and velocity of particle 1 are m_1 and $\mathbf{v}_1$, of particle 2 are m_2 and $\mathbf{v}_2$, and so on to particle n, this means that

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots + m_n\mathbf{v}_n \tag{4}$$

is a constant no matter what happens to the system. This is really helpful when you define coordinates so that your origin is at the center of mass. In that case, for a binary system, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$.

Relation between luminosity L, area A, and temperature T for a blackbody.

$$L = \sigma A T^4 \ . \tag{5}$$

Here $\sigma = 5.67 \times 10^{-8}$ J s^{-1} m^{-2} K^{-4} is the Stefan-Boltzmann constant. There's no reason at this stage to remember the value; the point is that it is a constant. We have usually used this equation for stars, where we assume that the star is a sphere. Since a sphere of radius R has area $4\pi R^2$, this means

$$L = \sigma 4\pi R^2 T^4 \ . \tag{6}$$

However, the first equation is more general.

Now let's try some problems.

1. The Sun has a radius of 7×10^8 m and a photospheric temperature of 5800 K. Its luminosity is 4.0×10^{26} J s^{-1} (we're using round numbers here to simplify). A very young white dwarf has a radius of 7×10^6 m and a photospheric temperature of 58000 K. What is its luminosity?

Answer:

You can do this problem the hard way or the easy way. The hard way would be to plug in the temperature and radius of the white dwarf into the equation for luminosity. That will get you the right answer, but it will also require that you put lots of numbers into your calculator, risking error. The easy way is to realize that the luminosity $L_\odot$ of the Sun and the luminosity $L_{\rm WD}$ of the white dwarf are related by

$$\begin{aligned} L_{\rm WD}/L_\odot &= \sigma 4\pi R_{\rm WD}^2 T_{\rm WD}^4/(\sigma 4\pi R_\odot^2 T_\odot^4) \\ &= (R_{\rm WD}/R_\odot)^2 (T_{\rm WD}/T_\odot)^4 \ . \end{aligned} \tag{7}$$

In our case $R_{\rm WD}/R_\odot = 0.01$, and $T_{\rm WD}/T_\odot = 10$, so $L_{\rm WD}/L_\odot = 1$, meaning $L_{\rm WD} = L_\odot = 4 \times 10^{26}$ J s^{-1}.

It's a good idea to carry things around symbolically (as T, R, or whatever) as long as possible, and only substitute the numbers in at the end. That saves effort.

2. You observe a binary star system. You find that the stars orbit in circles. Star A has a maximum radial velocity (also known as a line of sight velocity) of 20 km s^{-1}, and star B has a maximum radial velocity if 10 km s^{-1}. What is the ratio of masses between the stars?

Answer:

In a problem like this, your first thought should be to figure out which star is heavier, to check later against your specific answer. If star B is moving less, it's heavier (from Newton's laws). We can solve this by using conservation of momentum:

$$M_A v_A = M_B v_B \ . \tag{8}$$

Note that we have implicitly considered only the radial velocity component, which is why we haven't written this as a vector equation. This equation implies $M_A/M_B = v_B/v_A$, or $M_A/M_B = 0.5$ for our case. Therefore, star A is less massive than star B, as expected.

3. The Earth orbits the Sun at 30 km s^{-1}. How fast does an asteroid orbit the Sun if its motion is circular with a radius of 4 AU?

Answer:

Again, you could do this the hard way by directly putting in the numbers to the $v_{\rm circ} = \sqrt{GM/r}$ formula. But why do that? In this formula, M is $1\,M_\odot$ for both the Earth and the asteroid. G is also constant. Therefore, the only thing that varies is the radius r. We see that the speed is proportional to $1/\sqrt{r}$. Therefore, if r increases by a factor of 4, the speed is divided by 2. The asteroid moves at 15 km s^{-1}.

4. A planet is discovered around a $3\,M_\odot$ star, moving in a circle at the same speed that the Earth moves around the Sun. What is the orbital radius of the planet?

Answer:

Look at the formula for circular speed: $v_{\rm circ} = \sqrt{GM/r}$. We are given that $v_{\rm circ}$ is the same as it is for Earth. Also, G is the same (that's one nice thing about universal constants!). Therefore, we know that $\sqrt{M/r}$ must be the same for this planet as it is for Earth. But that also means that M/r is the same. Since M is three times larger, r must be as well. The distance is $r = 3$ AU. No calculator needed!

5. An amazing discovery is reported: a planet orbiting a black hole! The black hole has a mass of $100\,M_\odot$, the planet orbits in a circle, and it moves at 10 km s^{-1}. What is the orbital radius?

Answer:

Here you could break down and use the full equation :). However, if you happen to remember that the Earth's orbital speed is 30 km s^{-1}, and you notice that

$$r \propto M/v_{\rm circ}^2 \,, \tag{9}$$

you're set. Knowing that $r = 1$ AU for the Earth, you know that for the black hole planet M is 100 times larger and $v_{\rm circ}$ is 1/3 as large. Therefore, the orbital radius is 900 AU.

6. A spherical molecular cloud with a mass of $100\,M_\odot$ has a radius of 10^6 AU. What is the period of a particle in a circular orbit at the surface of this cloud?

Answer:

The Earth goes around the Sun in 1 year. Kepler's third law says that $P \propto \sqrt{r^3/M}$. Since $M = 100\,M_\odot$, that would decrease P by a factor of 10 for a fixed radius. But since $r = 10^6$ AU, that would increase P by a factor of $(10^6)^{3/2} = 10^9$ for a fixed mass. Therefore, the period is $10^9/10$ times the period of the Earth around the Sun, or 10^8 yr.