1. Stellar Rotation

The gravitational force $\vec{F}_g$ is related to the gravitational potential ϕ_g by

$$\vec{F}_g \;=\; -\nabla\phi_g \;, \tag{1}$$

where ∇ is the gradient operator, which in Cartesian coordinates is

$$\nabla \;=\; \frac{\partial}{\partial x}\,\hat{\mathbf{i}} \;+\; \frac{\partial}{\partial y}\,\hat{\mathbf{j}} \;+\; \frac{\partial}{\partial z}\,\hat{\mathbf{k}} \tag{2}$$

For a spherically symmetric mass distribution the gravitational potential is just

$$\phi_g \;=\; -\,\frac{G\;M_r}{r}\;, \tag{3}$$

but if rotation breaks that symmetry, ϕ_g will not be so simple. In any case, the potential is related to the mass distribution through the Poisson equation

$$\nabla^2\phi_g \;=\; 4\pi G\rho\;, \tag{4}$$

where ∇^2 is the Laplacian

$$\nabla^2 \;=\; \frac{\partial^2}{\partial x^2} \;+\; \frac{\partial^2}{\partial y^2} \;+\; \frac{\partial^2}{\partial z^2}\;. \tag{5}$$

Under the assumption that our star is rotating as a rigid body with an angular velocity ω, the velocity at any point is $v = \omega a$, where a is the distance from the axis of rotation (the z axis)(NB. a is not acceleration!). The centrifugal force is $\vec{F}_c = (v^2/a)\hat{\mathbf{a}}$, that is

$$\vec{F}_c \;=\; \omega^2 a\;\hat{\mathbf{a}} \;=\; \omega^2\;(x^2+y^2)^{1/2}\;\hat{\mathbf{a}}. \tag{6}$$

We see that this force can be obtained from the potential ϕ_c:

$$\phi_c \;=\; -\,\frac{1}{2}\omega^2\;(x^2+y^2)\;. \tag{7}$$

Thus the total force F acting on a mass element in the star is

$$\vec{F} \;=\; \vec{F}_g + \vec{F}_c \;=\; -\nabla\phi \quad \text{where} \quad \phi = \phi_g + \phi_c\;. \tag{8}$$

Now we see that $\nabla^2\phi_c \;=\; -\,2\omega^2$, so that the Laplacian of the total potential ϕ is

$$\nabla^2\phi \;=\; 4\pi G\rho \;-\; 2\omega^2\;. \tag{9}$$

We now introduce the more general form of the equation of hydrostatic equilibrium:

$$\nabla P \;=\; -\,\rho\;\nabla\phi\;. \tag{10}$$

You can see that in the absence of rotation, this just becomes the familiar $dP/dr = -\rho g$.

Now ∇P and $\nabla \phi$ are both vectors while ρ is a scalar. Thus ∇P must be parallel to $\nabla \phi$. If we move along a surface of constant ϕ, then P will also remain constant along this surface. Since P can only change with ϕ, the pressure can be written as a function of ϕ only: $P = P(\phi)$. But then, since

$$\frac{\nabla P(\phi)}{\nabla \phi} \;=\; -\,\rho\;, \qquad \text{it follows that} \qquad \rho \;=\; \rho(\phi)\;. \tag{11}$$

The equation of state for densities where we need only consider ideal gas pressure and radiation pressure is

$$P(\phi) \;=\; \frac{N_A k}{\mu}\,\rho(\phi)\,T \;+\; \frac{1}{3} a_{rad}\,T^4 \tag{12}$$

If we assume that the mean molecular weight μ is constant, or that it is a function of potential only, $\mu = \mu(\phi)$, then we see that all the variables in equation (12) other than T are functions of ϕ only. Thus we must also have $T = T(\phi)$. So we conclude that all the variables P,ρ, and T are constant along equipotential surfaces.

Next, let us consider the equation for *radiative energy transport*:

$$\vec{\mathcal{F}} \;=\; -\,\frac{4c\;a_{rad}}{3}\,\frac{T^3}{\kappa\rho}\,\nabla T \tag{13}$$

Now, since T is a function of ϕ only, we can write the gradient as

$$\nabla T \;=\; \frac{\partial \phi}{\partial x}\frac{dT}{d\phi}\,\hat{\mathbf{i}} \;+\; \frac{\partial \phi}{\partial y}\frac{dT}{d\phi}\,\hat{\mathbf{j}} \;+\; \frac{\partial \phi}{\partial z}\frac{dT}{d\phi}\,\hat{\mathbf{k}} \;=\; \frac{dT}{d\phi}\,\nabla\phi \tag{14}$$

Thus

$$\vec{\mathcal{F}} \;=\; \left\{-\frac{4c\;a_{rad}}{3}\,\frac{T^3}{\kappa\rho}\,\frac{dT}{d\phi}\right\}\nabla\phi \tag{15}$$

The expression is braces is a function of ϕ only, so we can write

$$\vec{\mathcal{F}} \;=\; f(\phi)\;\nabla\phi \qquad \text{where} \qquad f(\phi) = \left\{-\frac{4c\;a_{rad}}{3}\,\frac{T^3}{\kappa\rho}\,\frac{dT}{d\phi}\right\} \tag{16}$$

Now we turn to the equation of *thermal equilibrium*. The usual equation of stellar structure is

$$\frac{d\mathcal{F}}{dr} \;=\; \rho\epsilon\;\;, \tag{17}$$

where ϵ is the nuclear energy generation rate. (A more familiar form is obtained if we recall that $4\pi r^2 \mathcal{F} = L_r$). The 3-dimensional generalization of this equation involves the *divergence* of the radiative flux:

$$\nabla\cdot\vec{\mathcal{F}} \;=\; \rho\epsilon \qquad\qquad (\text{outside the core: }\; \nabla\cdot\vec{\mathcal{F}} \;=\; 0) \tag{18}$$

Now

$$\nabla\cdot\vec{\mathcal{F}} \;=\; \frac{\partial \mathcal{F}_x}{\partial x} \;+\; \frac{\partial \mathcal{F}_y}{\partial y} \;+\; \frac{\partial \mathcal{F}_z}{\partial z} \tag{19}$$

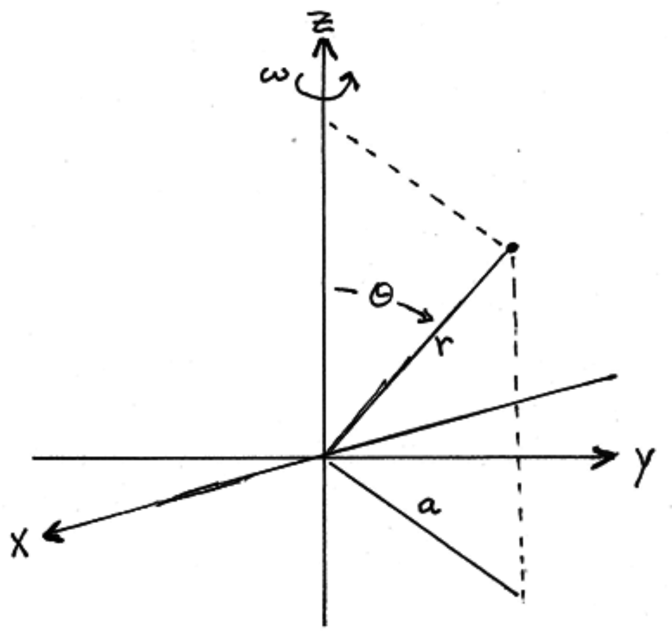

Fig. 1.— The coordinate system for a rotating star.

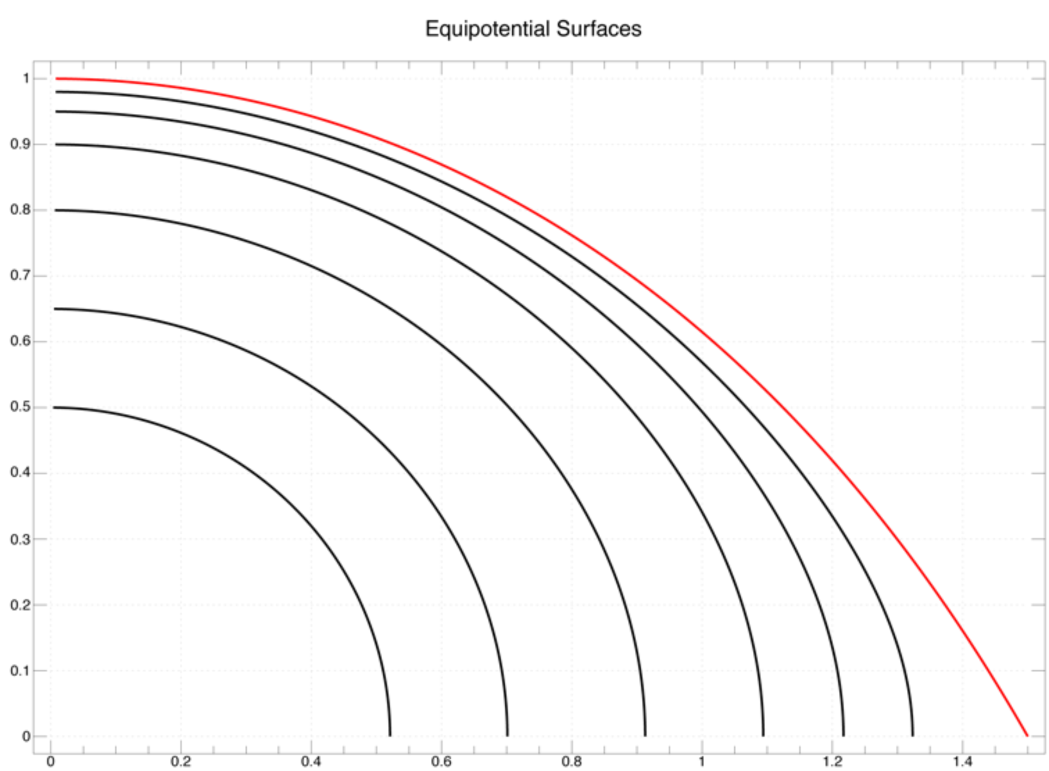

Fig. 2.— Equipotential surfaces of a rotating star (schematic).

Using equation (16) for the flux and noting that the x-component of $\nabla\phi$ is $\partial\phi/\partial x\ \hat{\mathbf{i}}$ etc., we see that

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \left[\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial x} + f\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)\right] + \left[\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial y} + f\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)\right] + \left[\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial z} + f\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)\right] \quad (20)$$

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \left[\left(\frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}\right)\frac{\partial \phi}{\partial x} \;+\; f\,\frac{\partial^2 \phi}{\partial x^2}\right] \;+\; \cdots \quad (21)$$

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \left[\frac{\partial f}{\partial \phi}\left(\frac{\partial \phi}{\partial x}\right)^2 \;+\; f\,\frac{\partial^2 \phi}{\partial x^2}\right] \;+\; \cdots \quad (22)$$

and combining the x,y and z components, we obtain

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \frac{\partial f}{\partial \phi}\,|\nabla\phi|^2 \;+\; f\,\nabla^2\phi \quad (23)$$

We now insert the value of the Laplacian from equation (9):

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \frac{\partial f}{\partial \phi}\,|\nabla\phi|^2 \;+\; f\left(4\pi G\rho - 2\omega^2\right) \quad . \quad (24)$$

With this expression for the divergence we return to equation (18) and obtain

$$\frac{\partial f}{\partial \phi}\,|\nabla\phi|^2 \;+\; f\left(4\pi G\rho - 2\omega^2\right) \;=\; \rho\epsilon \quad . \quad (25)$$

Examination of this equation shows that everything except for $\nabla\phi$ is a function of ϕ only, i.e., constant over a surface of constant ϕ. That is, if we travel along an equipotential surface while θ varies from 0 to π, everything except $\nabla\phi$ remains constant. On the other hand, $\nabla\phi$ *does* vary with θ. Looking at Fig. 2, we see that the surfaces are closer together at $\theta = 0$ and further apart at $\theta = \pi/2$. This means that $\nabla\phi$ is greater in the polar direction (along the z-axis) than in the equatorial direction. The only way to reconcile these restrictions is if

$$\frac{df}{d\phi} \;=\; 0 \qquad\qquad \text{which implies that} \qquad f \;=\; \text{constant} \quad (26)$$

As a result we see that

$$\rho\epsilon \;=\; f\left(4\pi G\rho - 2\omega^2\right) \quad (27)$$

And since f is constant we can write this as

$$\epsilon \;=\; \textit{constant}\left(1 - \frac{\omega^2}{2\pi G\rho}\right) \quad (28)$$

This result is known as *von Zeipel's paradox* (von Zeipel 1924). For how can the rate of nuclear energy generation ϵ be directly determined by the angular velocity of the star's rotation? This is not possible, so what we have proved is that a uniformly rotating star cannot be in radiative thermal equilibrium.

What may happen is that, in regions outside the core where we should have $\nabla \cdot \vec{\mathcal{F}} \; = \; 0$, we will instead have

$$\nabla \cdot \vec{\mathcal{F}} \;=\; \frac{\partial f}{\partial \phi}\,|\nabla\phi|^2 \;+\; f\left(4\pi G\rho - 2\omega^2\right) \;\neq\; 0\;. \tag{29}$$

The divergence of the radiative flux will be positive in some directions and negative in others, which will cause heating or cooling, which in turn may lead to slow currents within the star which will balance this excess heating or cooling. These currents are known as Eddington-Sweet currents. See Fig. 3. Now the currents will not be very fast, so the equation of hydrostatic equilibrium (equation 10) will be obeyed closely. And all the steps in the derivation will hold up to equation (16), but now we can't have $f(\phi)$ strictly constant from one ϕ to the next. But consider some equipotential surface ϕ_0 just under the surface. Over this surface

$$\vec{\mathcal{F}} \;=\; f(\phi_0)\;\nabla\phi \;=\; f_0\;\vec{\mathbf{g}} \tag{30}$$

and this must be continuous with the radiation escaping from the atmosphere. Here recall that the gradient of the total potential ϕ is just $\vec{\mathbf{g}}$, the *effective* gravity, i.e., the gravity minus the centrifugal acceleration. This behavior ($\vec{\mathcal{F}} \propto \vec{\mathbf{g}}$) is called "gravity darkening" since the stellar surface near the equator, where the effective gravity is small, will have a reduced flux and will thus be "darker". The scalar form of this relation,

$$\mathcal{F} \;\propto\; g \qquad \Longrightarrow \qquad T_{eff} \;\propto\; g^{1/4} \tag{31}$$

is referred to as "von Zeipel's law". Thus a rapidly rotating star will have a surface temperature which varies with latitude, highest at the pole and lowest at the equator.

2. The Shape of Rotating Stars

Going back to the total potential

$$\phi \;=\; \phi_g \;-\; \frac{1}{2}\omega^2\;a^2 \tag{32}$$

where $a \;=\; (x^2 \,+\, y^2)^{1/2}$, the distance from the z-axis. Most stars are rather centrally condensed, so we can justify adopting a *Roche model*, that is, a model with all the mass concentrated in the center. Then we can approximate ϕ_g by the potential of a spherical distribution

$$\phi_g \;=\; -\,\frac{GM_r}{r} \tag{33}$$

so at the surface the potential is

$$\phi \;=\; -\,\frac{GM}{r} \;-\; \frac{1}{2}\omega^2\;a^2 \tag{34}$$

and the local gravity is

$$\vec{\mathbf{g}} \;=\; -\,\nabla\phi \;=\; -\,\frac{GM}{r^2}\;\hat{\mathbf{r}} \;+\; \omega^2 a\;\hat{\mathbf{a}} \tag{35}$$

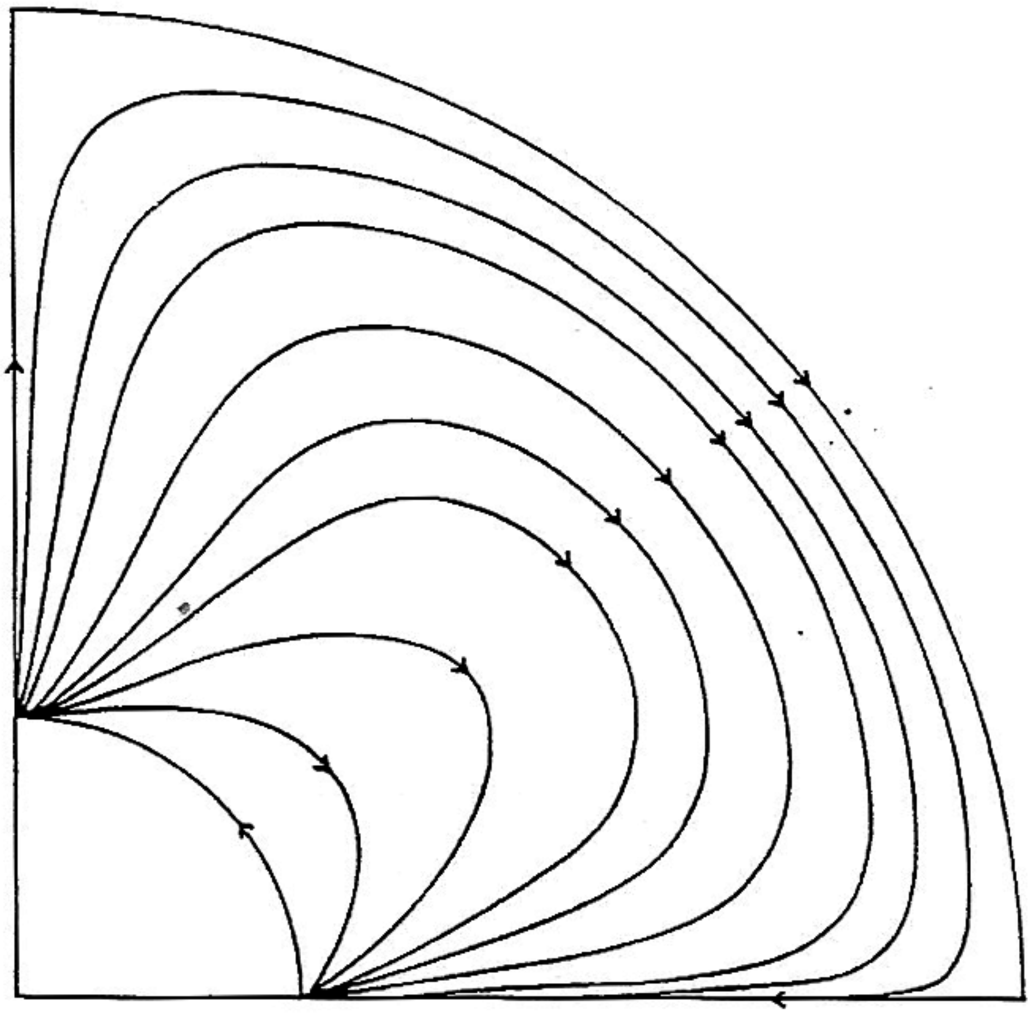

Fig. 3.— Eddington-Sweet currents outside a convective core.

At the pole $a = 0$ while at the equator $a = r$.

What is the break-up rotational velocity of the star? We seek the critical ω_c such that $g = 0$ at the equator. Thus, if r_e is the equatorial radius,

$$g \;=\; 0 \;=\; -\,\frac{GM}{r_e^2} \;+\; \omega_c^2 r_e \tag{36}$$

$$\omega_c^2 \;=\; \frac{GM}{r_e^3} \qquad \Longrightarrow \qquad \omega_c \;=\; \left(\frac{GM}{r_e^3}\right)^{1/2} \tag{37}$$

But we don't know what r_e is, since the star stretches out! Now the pole is on the same potential surface as the equator. Thus, if r_p is the polar radius,

$$-\frac{GM}{r_p} \;=\; -\,\frac{GM}{r_e} \;-\; \frac{1}{2}\omega_c^2\, r_e^2 \tag{38}$$

$$-\frac{GM}{r_p} \;=\; -\,\frac{GM}{r_e} \;-\; \frac{1}{2}\left(\frac{GM}{r_e^3}\right)\, r_e^2 \;=\; -\,\frac{3}{2}\frac{GM}{r_e} \tag{39}$$

Thus we see that

$$r_e \;=\; \frac{3}{2}\, r_p \qquad\qquad \text{for} \qquad \omega = \omega_c \tag{40}$$

Further, the rotational velocity of the equator at break-up is

$$v_c \;=\; r_e\,\omega_c \;=\; \left(\frac{GM}{r_e}\right)^{1/2} \;=\; \left(\frac{2}{3}\,\frac{GM}{r_p}\right)^{1/2} \tag{41}$$

Now we may be tempted to argue that r_p won't change with a star's rotation – after all, the gravity along the z-axis doesn't change. (This sounds reasonable, but is only approximately true.) We will assume it's OK.

So what is the break-up velocity of the sun? Put $r_p = R_\odot$ and $M_\odot$ into equation (41) and we find $v_c = 357$ km/s, which corresponds to a period of $P = 3\pi\; r_p/v_e = 5.1$ hr. Actually, the sun's rotation is much slower, $v_e = 2$ km/s, $P = 25$ days. Over 4 billion years, the sun has lost most of its angular momentum to the solar wind, to which it is magnetically coupled.

Here are some examples of break-up velocities:

Type	$M/M_\odot$	v_c km/s	Period
Main Sequence	1	357	5.1 hr
	3	456	7.3 hr
	5	507	8.9 hr
	10	584	11.6 hr
White Dwarf	1.0	4000	13.1 sec
Neutron Star	1.4	100000	0.001 sec

If we define a scaled radius $x(\theta, w) = R(\theta, \omega)/R_p$, then it is easy to see that the requirement that all points on the star lie on the same equipotential surface as the pole (where

$x = 1$) leads to a cubic equation for x:

$$\frac{4}{27} (w \ \sin\theta)^2 \ x^3 \ - \ x \ + \ 1 \ = \ 0 \ \ , \tag{42}$$

where w is the angular velocity in terms of the critical velocity, $w = \omega/\omega_c$. Thus the shape function x may be written, using the trigonometric form of the solution of the cubic equation, as:

$$x(\theta, w) \ = \ \frac{3}{w \ \sin\theta} \ \cos\left\{\frac{1}{3}\left[\pi + \arccos(w \ \sin\theta)\right]\right\} \tag{43}$$

Here are some equatorial to polar ratios for various values of w:

$w = \omega/\omega_c$	$x(\pi/2, w) = R_e/R_p$
0.5	1.042
0.8	1.141
0.9	1.216
0.95	1.281
0.99	1.390
1.0	1.5

The next figure shows the shapes obtained from this equation for various values of w. To obtain the value of the effective gravity, we must remember that the two components in equation (35) are not parallel. If we take a coordinate axis η perpendicular to the z-axis in the plane of our point, this is the direction of the centripetal force. We can resolve the first term into components:

$$-\frac{GM}{r^2} \ \hat{\mathbf{r}} \ = \ - \ \frac{GM}{x^2 \ R_p^2} \left[\sin\theta \ \hat{\eta} \ + \ \cos\theta \ \hat{\mathbf{z}}\right] \tag{44}$$

Thus the effective gravity can be written

$$\vec{\mathbf{g}} \ = \ \left[\frac{-GM}{x^2 \ R_p^2} \ + \ \omega^2 \ x \ R_p\right] \sin\theta \ \hat{\eta} \ + \ \left[\frac{-GM}{x^2 \ R_p^2}\right] \cos\theta \ \hat{\mathbf{z}} \tag{45}$$

and the magnitude of the effective gravity as a function of θ follows from $g = |\vec{\mathbf{g}}| = (\vec{\mathbf{g}} \cdot \vec{\mathbf{g}})^{1/2}$:

$$g \ = \ \left[\frac{GM}{x^2 \ R_p^2}\right] \left\{\cos^2\theta \ + \ \sin^2\theta \left[1 \ - \ \omega^2 \ \frac{x^3 \ R_p^3}{GM}\right]^2\right\}^{1/2} \tag{46}$$

From equations (37) and (39) we see that $\omega_c^2 = (2/3)^3 \ GM/R_p^3$, and using $\omega^2/\omega_c^2 = w^2$ we obtain

$$g \ = \ \left(\frac{GM}{R_p^2}\right) \ \frac{1}{x^2} \ \left\{\cos^2\theta \ + \ \sin^2\theta \left[1 \ - \ \frac{8}{27} \ x^3 \ w^2\right]^2\right\}^{1/2} \quad . \tag{47}$$

The figure shows the variation of gravity with latitude for a number of rotational velocities. As mentioned above, the surface brightness is proportional to the gravity, while the surface temperature varies as the fourth root of the local effective gravity. These effects have in fact been measured in nearby rapidly rotating stars such as Regulus and Vega.

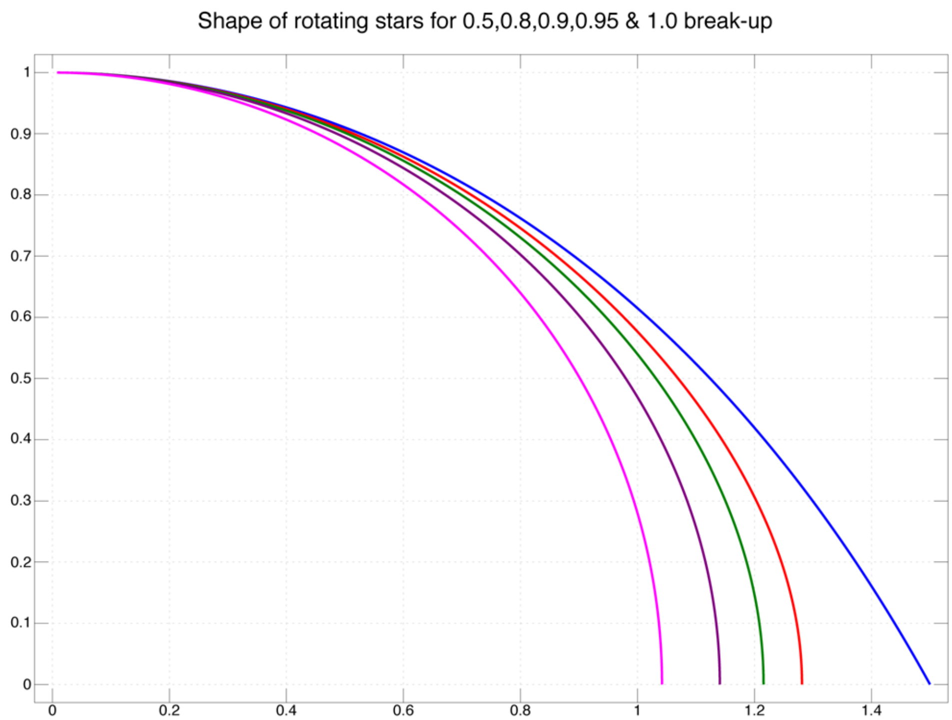

Fig. 4.— The shapes of rapidly rotating stars from equation (42).

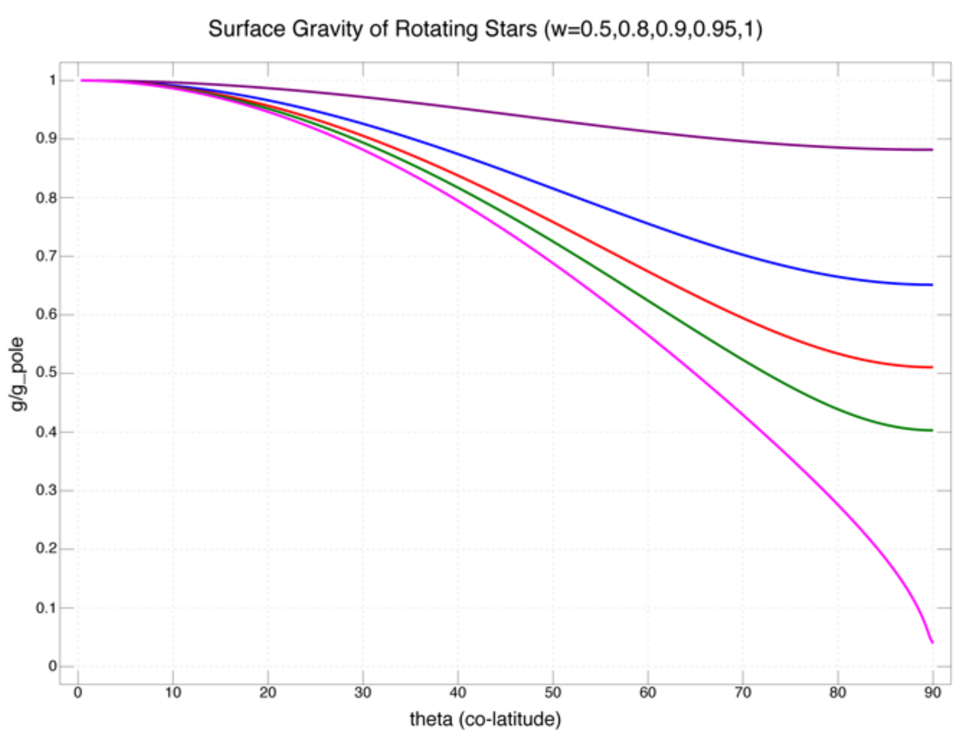

Fig. 5.— The variation of surface gravity for various rotational velocities.

3. The Surface Area and Temperature of a Rotating Star

While we might adopt the *relative* variation of surface temperature with latitude from von Zeipel's "law" (equation 31), we do not know the zero point: e.g., what is the temperature of the pole? Compare the rotating star to a spherical non-rotating star of the same mass. If the pole has the same temperature as the non-rotating star, then as the temperature drops towards the equator, less flux will be radiated and the luminosity of the rotating star will be lower than that of its static counterpart. (The area of the rotating star will be larger than that of the spherical star, but as we will see, that is not enough to compensate for the gravity darkening.) Lets look at this for the Roche model. The coordinates and angles for our discussion are shown in Figs. 6 and 7. Any cross-section through the rotation axis will slice out a plane $Z\eta$ and we see that for any co-latitude θ, the unit normal to the surface, $\hat{n}$, makes an angle δ with the Z-axis. What is the angle δ ? We look at the η and Z components of the local gravity (from equation 44):

$$\vec{\mathbf{g}} \;=\; \left[\frac{-g_p}{x^2} \;+\; \omega^2\; x\; R_p\right]\sin\theta\;\hat{\eta} \;+\; \left[\frac{-g_p}{x^2}\right]\cos\theta\;\hat{\mathbf{z}} \quad , \tag{48}$$

where $g_p = GM/{R_p}^2$ is the magnitude of the gravity at the pole. Clearly the ratio of the η component to the Z component is $\tan\delta$. Thus

$$\tan\delta \;=\; \left\{1 \;-\; \frac{\omega^2 x^3 R_p}{g_p}\right\}\;\tan\theta \tag{49}$$

Now the last term in the braces is just

$$\frac{\omega^2 x^3 R_p}{g_p} \;=\; \omega^2\frac{R_p^3 x^3}{GM} \;=\; \frac{\omega^2}{\omega_c^2}\frac{8}{27}x^3 \;=\; \frac{8}{27}x^3 w^2 \tag{50}$$

(see p 8) so that we obtain

$$\tan\delta \;=\; \left\{1 \;-\; \frac{8}{27}x^3 w^2\right\}\;\tan\theta \quad . \tag{51}$$

Note that when taking the arctan to obtain δ, we need to make sure δ is in the same quadrant as θ. (See Fig. 8.)

Now to obtain the surface area of the star, we integrate‘ over the range $0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$. The element of area in the radial direction $\hat{r}$ is just

$$dA' \;=\; R(\theta,w)^2\; d\phi\;\sin\theta\; d\theta \;=\; R_p^2\;\; x(\theta,w)^2\; d\phi\;\sin\theta\; d\theta \tag{52}$$

with $x(\theta,w)$ from equation (43). There is no ϕ dependence, so we can do that integration. Now the stellar surface element dA is inclined to the radial surface element dA' by the angle $\theta-\delta$, so dA projects onto dA' by a factor of $\hat{n}\cdot\hat{r} = \cos(\theta-\delta)$. Thus we must divide by this factor:

$$A \;=\; \int\!\!\int dA \;=\; 4\pi\; R_p^2\;\; \frac{1}{2}\int_0^{\pi} x(\theta,w)^2\;\frac{\sin\theta}{\cos(\theta-\delta)}\; d\theta \tag{53}$$

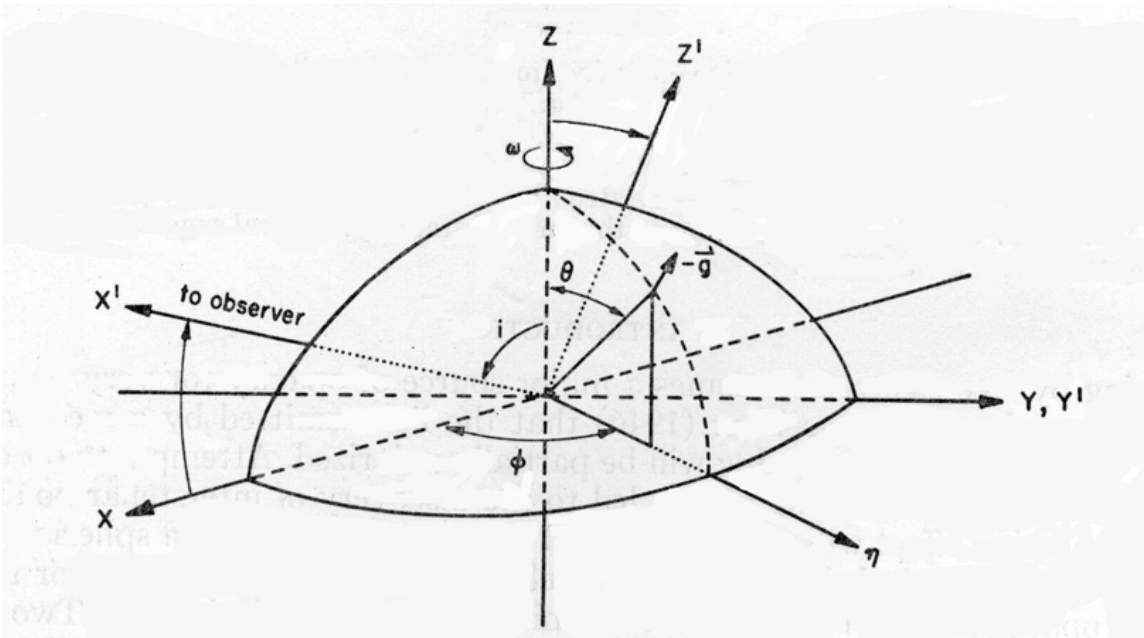

Fig. 6.— The coordinate systems defining the angles used in the text.

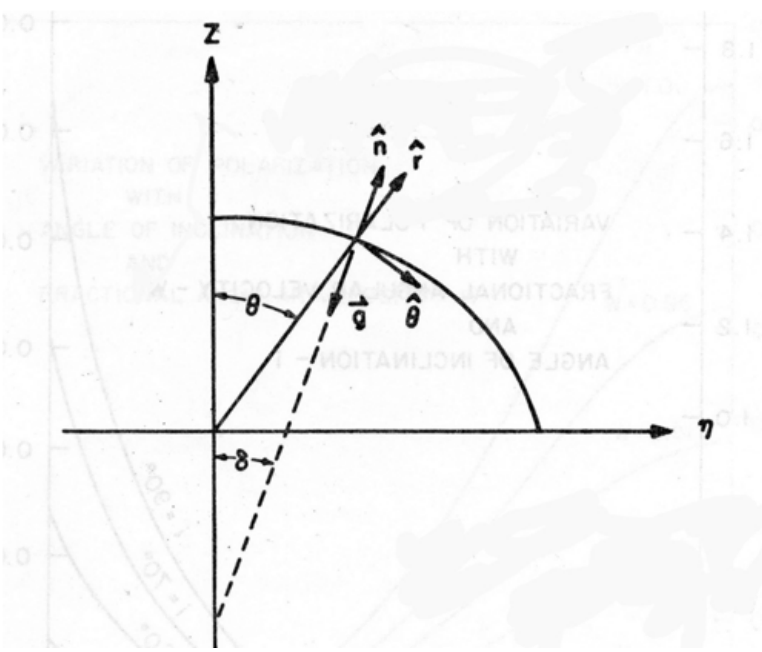

Fig. 7.— Cross-section of star in the $Z\eta$ plane, defining angle δ.

We can then carry out the integration over θ numerically to find the surface area A of the star. We provide some results in the table below.

The luminosity of our rotating star will be given by the flux from each surface element integrated over the star. From equation (31), $\mathcal{F} = \sigma T_{eff}^4(\theta) \propto g(\theta)$, but von Zeipel's relation does not give us the constant of proportionality. If, however, we assign the temperature at one point, e.g., the pole, then we can write

$$L \;=\; \int\!\!\int \mathcal{F}\; dA \;=\; 2\pi\; R_p^2\; \sigma T_{pole}^4 \;\int_0^\pi x(\theta,w)^2\; \frac{\sin\theta}{\cos(\theta-\delta)}\; \left[\frac{T(\theta)}{T_{pole}}\right]^4\; d\theta \qquad (54)$$

Now

$$\left[\frac{T(\theta)}{T_{pole}}\right]^4 \;=\; \frac{g(\theta)}{g_{pole}} \;=\; \frac{1}{x^2}\; \left\{\cos^2\theta \;+\; \sin^2\theta \left[1 \;-\; \frac{8}{27}\; x^3\; w^2\right]^2\right\}^{1/2} \qquad (55)$$

by equation (47). The x's cancel leaving us with

$$L \;=\; 4\pi\; R_p^2\; \sigma T_{pole}^4 \quad \frac{1}{2}\int_0^\pi \frac{\sin\theta}{\cos(\theta-\delta)}\; \left\{\cos^2\theta + \sin^2\theta \left[1 \;-\; \frac{8}{27}\; x^3\; w^2\right]^2\right\}^{1/2} d\theta \qquad (56)$$

The leading term, $4\pi\; R_p^2\; \sigma T_{pole}^4$, is just the luminosity of a spherical star of radius R_p and surface temperature T_{pole}. The remaining integral thus gives the ratio of the luminosity of a rotating star with polar radius R_p to a non-rotating star of radius R_p and surface temperature T_{pole}. Some results are given in the table below. We see that the total luminosity of the rotating stars will be substantially less than the non-rotating counterpart if the polar temperature is held to that of the non-rotating star.

On the other hand, we might consider the proposition that the luminosity of the star (which we might think is not much affected by the distortion of the outer envelope) should remain constant. Now if we simply increase the temperature for all θ by a factor $(L_{sphere}/L)^{1/4}$, then the luminosity will remain unchanged. This factor, which is just T_{pole}/T_{sphere}, is given in the last column.

$w = \omega/\omega_c$	A/A_{sphere}	L/L_{sphere}	T_{pole}/T_{sphere}
0.5	1.056	0.9465	1.0139
0.8	1.186	0.8375	1.0453
0.9	1.282	0.7705	1.0674
0.95	1.363	0.7222	1.0848
0.99	1.485	0.6645	1.1076
1.0	1.577	0.6394	1.1183

We should emphasize that we are simply assuming a constant polar radius and luminosity. This is just a toy model; there are much more complex studies which do in fact find modest decreases in luminosity and changes in polar radius. Further, we know that the assumption of rotation with a uniform angular velocity ω is violated. We will return to this later. For now, we want a simple model to investigate the effects of rotation on the variation of apparent luminosity with viewing angle and the degree of polarization of the star's radiation resulting from the star's non-spherical shape.

3.1. Beyond von Zeipel: A Improved Gravity Darkening Relation

We noted above that the von Zeipel law (eq. 31) neglects effects of the Eddington-Sweet curents. Observations of nearby rapid rotators with stellar interferometers have found that the von Zeipel law overestimates the temperature contrast between the pole and equator. While modeling of rotating stars is very difficult (e.g., Rieutord and Espinosa Lara 2009), some models have been computed and the resulting gravity darkening differs from the von Zeipel law.

In light of these results, Espinosa Lara and Rieutord (2011) have presented a method of computing an improved gravity darkening law based on the assumption that the flux throughout the star is antiparallel to the effective gravity. They show that the the darkening law can be obtained by solving one trancendental equation. They define

$$\tau \;=\; \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta \;+\; \cos\theta \;+\; \ln\tan\frac{\theta}{2} \tag{57}$$

where θ is the co-latitude and $\tilde{r} = r/R_{eq}$. They assume that the star is centrally condensed so that the shape will be given by the Roche model. In terms of $x(\theta, w)$ (eq 43), $\tilde{r} = x(\theta, w)/x(\pi/2, w)$. In these equations, the angular velocity ω is defined as Ω/Ω_k, where Ω_k is the Keplerian angular velocity at the star's equator, $(GM/R_e^3)^{1/2}$. Note that ω is not the same as w: w is defined in terms of the break-up velocity of the star, while ω is in terms of the Keplerian velocity at the equator of the star which is in gereral not at break-up velocity. In fact,

$$\omega \;=\; \left[\frac{2}{3}\; x(w, \frac{\pi}{2})\right]^{3/2} \; w \tag{58}$$

Having evaluated τ (eq 57), Espinosa Lara and Rieutord show that we only need solve

$$\tau \;=\; \cos\Theta \;+\; \ln\tan\frac{\Theta}{2} \tag{59}$$

for Θ. Then the deviation from the von Zeipel law is given by the the quatity

$$F_\omega(\theta) \;=\; \frac{\tan^2\Theta}{\tan^2\theta} \qquad \text{so that} \qquad T_{eff}(\theta) \;\propto\; \left(F_\omega(\theta)\; g(\theta)\right)^{1/4} \tag{60}$$

This expression for F_ω breaks down at the pole and equator; for these points we have

$$F_\omega(\theta = 0) \;=\; e^{\frac{2}{3}\omega^2\tilde{r}^3} \qquad \text{and} \qquad F_\omega(\theta = \pi/2) \;=\; (1 - \omega^2\tilde{r}^3)^{-2/3} \tag{61}$$

While the results for this formulation approaches the von Zeipel law for low angular velocities (w $\leq$ 0.5), it differs substantially from von Zeipel for high angular velocities. For example, for w=0.6,0.8,0.9,0.95 and 0.99 (ω = 0.3586,0.5306,0.6565,0.7499 and 0.8828), the T_{eq}/T_p ratios for the Espinosa Lara and Rieutord formulation are, respectively, 0.9414, 0.8825, 0.8315, 0.7887 and 0.7126. For the Von Zeipel law, the corresponding T_{eq}/T_p ratios are 0.9365, 0.8620, 0.7878, 0.7186 and 0.5814. Also, see Fig. 10.

4. The Stellar Flux as a Function of the Viewing Angle

In the last section we considered the total luminosity of the rotating star and found how an increase in the polar temperature could keep the luminosity constant for all rates of rotation. But a rotating star will have a different apparent brightness when viewed at different angles of inclination (here we define the angle of inclination i as the angle between the observers line-of-sight and the axis of rotation). To this end, we need to consider the intensity radiated by each patch of stellar surface in the direction of the observer. The intensity radiated, $I(\mu)$, will be a function of the local temperature and of μ, the cosine of the angle between the observer and the normal to the surface $\hat{n}$. To obtain the observed flux, we must multiply $I(\mu)$ by the foreshortening of the surface element dA, which is just μ. Thus the flux seen by the observer will be

$$F(i) \;=\; \int\!\!\int I(\mu)\mu \; dA \;=\; R_p^2 \;\int_{-\pi/2}^{\pi/2}\int_0^{\pi} I(|\mu|)|\mu| \; \frac{x^2 \;\sin\theta \; d\theta}{\cos(\theta-\delta)} \; d\phi \tag{62}$$

Now we cannot do the integration over ϕ by inspection since μ is a function of ϕ as well as θ. Let us consider μ. It is the cosine of the angle between the normal to the stellar surface, $\hat{n}$, and the direction to the observer, which is the X' axis (see Fig. 6). That is $\mu = \hat{x'} \cdot \hat{n}$. Now

$$\hat{n} \;=\; \sin\delta\;\cos\phi\;\hat{x} \;+\; \sin\delta\;\sin\phi\;\hat{y} \;+\; \cos\delta\;\hat{z} \tag{63}$$

and

$$\hat{x'} \;=\; \sin i\;\hat{x} \;+\; \cos i\;\hat{z} \tag{64}$$

so that

$$\mu \;=\; \hat{x'}\cdot\hat{n} \;=\; \sin i\;\sin\delta\;\cos\phi \;+\; \cos i\;\cos\delta \tag{65}$$

A comment is needed regarding the limits of integration. By symmetry, we need only integrate ϕ over the interval $[0, \pi/2]$. The θ integration seems more complex, since, depending upon the inclination i, part of one hemisphere will be hidden (when the angle between $\hat{x'}$ and $\hat{n}$ is greater than $\pi/2$) while for the opposite hemisphere we will see over the pole (regions where $\pi/2 \le \phi \le 3\pi/2$). But a little consideration shows that the hidden region corresponds exactly to the newly revealed region over the pole, and we can always integrate θ from 0 to π, *provided we take the absolute value of* μ.

The following table shows the values of the flux $F(i)$ for a number of rotational velocities for 5 angles of inclination. The values are relative to a total luminosity of unity. This calculation used Eddington's first approximation for the the frequency integrated radiation:

$$I(\mu) \;=\; \frac{3}{4}\,F\,\left(\mu \;+\; \frac{2}{3}\right) \;\;, \tag{66}$$

where the flux F varies with latitude as $F = (g(\mu)/g_{pole})(L_{sphere}/L)$. The stellar surface was covered by over 1.6×10^6 points.

$w = \omega/\omega_c$	$i = 0^o$	$i = 15^o$	$i = 30^o$	$i = 45^o$	$i = 60^o$	$i = 75^o$	$i = 90^o$
0.5	1.0745	1.0670	1.0466	1.0187	0.9907	0.9702	0.9627
0.8	1.2626	1.2365	1.1648	1.0662	0.9672	0.8946	0.8680
0.9	1.4114	1.3707	1.2588	1.1041	0.9484	0.8345	0.7929
0.95	1.5401	1.4871	1.3405	1.1369	0.9320	0.7824	0.7279
0.99	1.7260	1.6556	1.4595	1.1842	0.9062	0.7071	0.6343
1.0	1.8225	1.7435	1.5222	1.2082	0.8945	0.6682	0.5861

We see that for this model, a star rotating at 95% break-up will appear twice as bright when viewed from the pole as when viewed from the equator. This effect should be considered when rapidly rotating stars are placed on the H-R diagram.

5. The Net Polarization of Radiation from Rotating Stars

In stellar atmospheres where the scattering (either by free electrons or by molecules) is a substantial fraction of the opacity, the emergent radiation will be partially polarized. For isolated spherical stars, such polarization will cancel out and thus be unobservable, unless the disk is partially masked by occultation or the stellar disk can be resolved. However, if the star is rapidly rotating, the symmetry is broken and a net polarization will result. We would like to know if this polarization ever reaches an observable magnitude.

5.1. Results for Pure Scattering Atmospheres

The first such calculations were made by George Collins and myself when I was a graduate student (Harrington & Collins, 1968, hereinafter HC). For lack of any more realistic atmosphere model including polarization, we used the solution of Chandrasekhar for a pure scattering atmosphere (Chandrasekhar: 1946, 1960) to provide the Stokes parameters $I(\mu)$ and $Q(\mu)$ (we actually used the equivalent parameters I_l and I_r). In this section I will repeat that calculation and then proceed to a calculation using a realistic model atmosphere.

When polarization is included, the radiation emerging from the atmosphere can be characterized by the Stokes parameters I, Q, U, V. Since the scattering we consider does not produce circular polarization, $V = 0$. Furthermore, if the reference plane for the polarization is the meridian plane (i.e., referenced to the normal to the surface), then by symmetry we must have $U = 0$. Thus we only need $I(\mu)$ and $Q(\mu)$ for the emergent radiation. Chandrasekhar tabulates the degree of polarization, $P(\mu)$; $Q(\mu) = P(\mu)I(\mu)$. To combine the polarization from different patches on the stellar surface, we must express this local polarization with reference to a common axis, which will be the projection of the stellar rotation axis (the Z' axis of Fig. 6). Let the projection of the local normal $\hat{n}$ onto the $Z' - Y'$ plane make an angle ξ with the Z' axis. Then we need to rotate the local emergent $Q(\mu)$ clockwise (as seen traveling with the ray) through the angle ξ. This rotation gives rise to a non-zero Stokes U'. The equations are (see $http://www.astro.umd.edu/\sim jph/notes3.pdf$):

$$I'(\mu) \;=\; I(\mu) \;\;, \quad Q'(\mu) \;=\; Q(\mu)\;\cos(2\xi) \;\;, \quad U'(\mu) \;=\; Q(\mu)\;\sin(2\xi) \;\;. \tag{67}$$

What is this angle ξ ?

The local normal $\hat{n}$ is given by equation (63). We want to express this in terms of the observer's coordinates X'Y'Z':

$$\hat{x} \;=\; \sin i\;\hat{x'} \;-\; \cos i\;\hat{z'} \tag{68}$$

$$\hat{y} \;=\; \hat{y'} \tag{69}$$

$$\hat{z} \;=\; \cos i\;\hat{x'} \;+\; \sin i\;\hat{z'} \tag{70}$$

Thus the local normal can be written

$$\hat{n} = (\sin i\;\sin\delta\;\cos\phi + \cos i\;\cos\delta)\hat{x'} \;+\; \sin\delta\;\sin\phi\;\hat{y'} \;+\; (\sin i\;\cos\delta - \cos i\;\sin\delta\;\cos\phi)\hat{z'} \tag{71}$$

Now the $\hat{x'}$-component is just the component along the line-of-sight, which is μ, in agreement with equation (65). Further, the tangent of the angle between the Z' axis and the projection of $\hat{n}$, $\tan\xi$, is the ratio of the Y' component to the Z' component:

$$\tan\xi \;=\; \frac{\sin\delta\;\sin\phi}{\sin i\;\cos\delta \;-\; \cos i\;\sin\delta\;\cos\phi} \tag{72}$$

The tabulated values of $I(\mu)$ are such that the flux $F = 2\int_0^1 I(\mu)\mu\;d\mu = 1$. These values must be scaled up to the required flux $\pi F = \sigma T^4(\theta)$. We can write the scaled Stokes parameters as

$$\mathcal{I}(\theta,\phi) \;=\; \sigma T^4(\theta)\;I(|\mu|) \;\;, \quad \mathcal{Q}(\theta,\phi) \;=\; \sigma T^4(\theta)\;\cos(2\xi)\;Q(|\mu|) \;\;. \tag{73}$$

Thus the Stokes parameters integrated over the stellar surface will be

$$S(i) \;=\; 2R_p^2 \int_0^{\pi/2}\int_0^{\pi} \mathcal{S}(\theta,\phi)\;|\mu|\;\frac{x^2\sin\theta}{\cos(\theta-\delta)}\;d\phi\;d\theta \;\;, \tag{74}$$

where $\mathcal{S}$ represents both $\mathcal{I}$ and $\mathcal{Q}$ and $S(i)$ those parameters integrated over the stellar surface. Were we to include the Stokes $\mathcal{U}$, and integrate over the full range $-\pi/2 \le \phi \le \pi/2$, because of symmetry, the resulting $U(i)$ would be exactly zero. $\mathcal{Q}$ is symmetric between the right and left hemispheres, so we may just take twice the integral over $0 \le \phi \le \pi/2$. With x from eqn. (43), δ from eqn. (51), μ from eqn (65), ξ from eqn (72), and $T(\theta)$ from eqn (55), we can then carry out the numerical integration.

We set T_{pole} to keep the total luminosity constant for all w (see the discussion on p 12). We used 901 values of ϕ and 1801 values of θ to cover the quarter-surface of the star. Fig. 9 shows the ratio of the flux seen by the observer to the flux from a non-rotating spherical

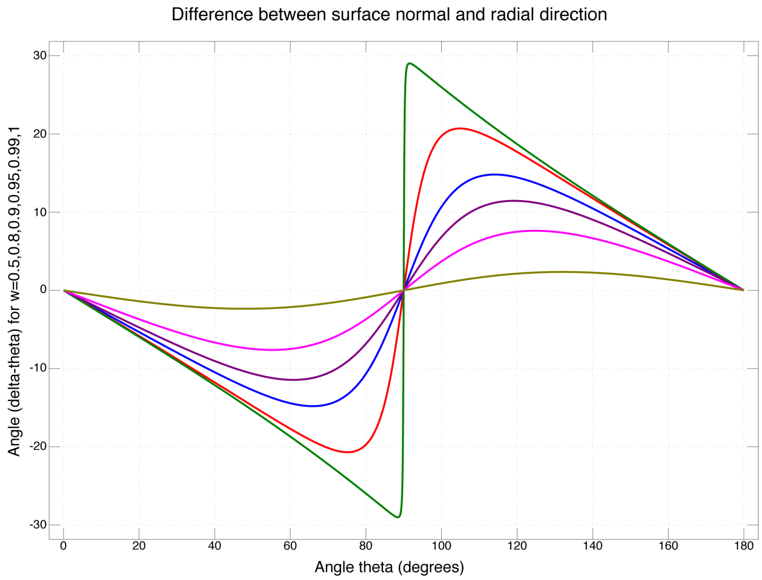

Fig. 8.— The difference $(\delta - \theta)$ as a function of θ for various w.

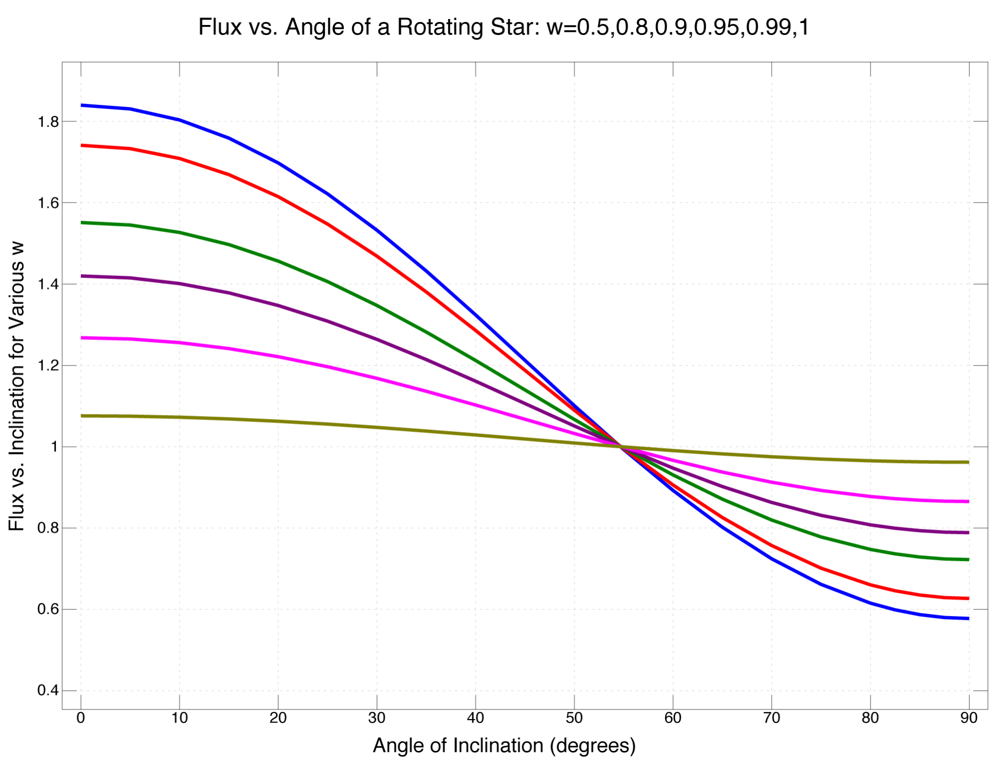

Fig. 9.— The flux as a function of inclination angle for various rotation velocities. These results are for the frequency-integrated radiation from a pure scattering atmosphere.

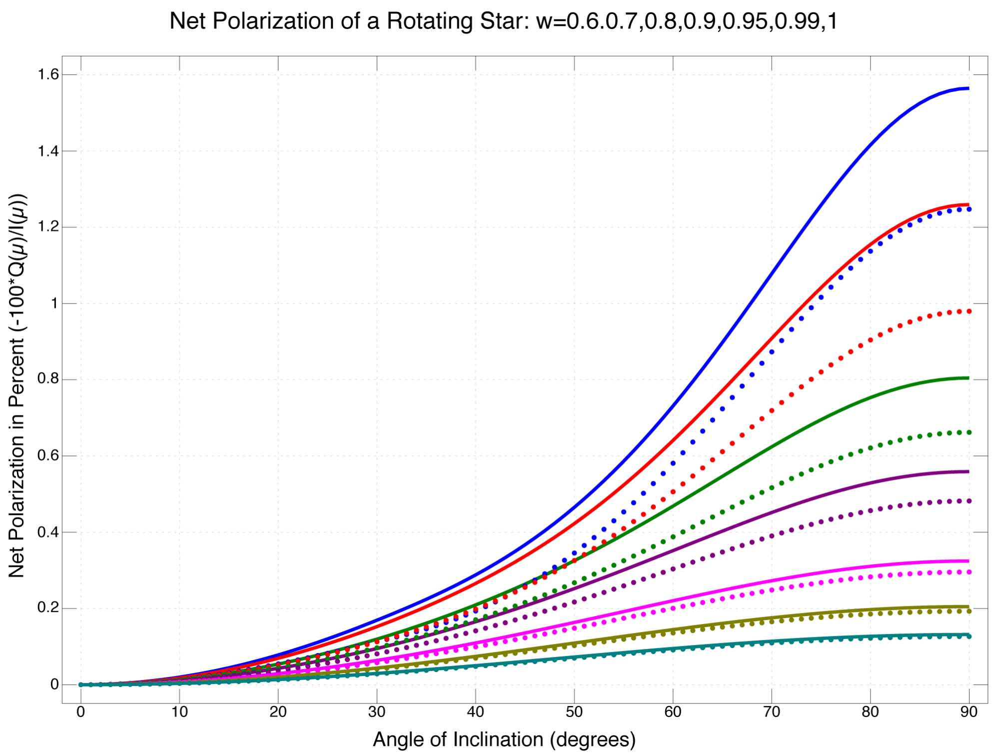

Fig. 10.— The percent net polarization vs inclination for various rotation velocities.
These results are for the frequency-integrated radiation from a pure scattering atmosphere. Solid lines are von Zeipel's law while dotted lines are for the flux law of Espinosa Lara and Rieutord (see §3.1).

star. Fig. 10 shows the percentage of polarization, $p(i) = -Q(i)/I(i)$. We see that the net polarization for rapidly rotating stars *with a pure scattering atmosphere* reaches levels of about 1 percent.

If we compare these results to those in HC, we find that they do not agree in detail. The results shown in Fig. 4 of HC are generally 30-50% higher than those shown here.

The equations we have developed here are equivalent to those used by HC. At present, I do not understand the lack of agreement. The code used in the earlier calculations are are of course lost in the mists of time, so it is not easy to track down the source of the disagreement.

5.2. Results for Realistic Model Atmospheres

Next, we would like to consider the results using the more realistic models for stellar atmospheres which we have tabulated above. For a given rotational velocity, parameterized by $w = \omega/\omega_{break-up}$, there will be a well defined variation of surface temperature and of surface gravity with θ (i.e. stellar co-latitude). Consider a non-rotating B1 V star, which may have a surface temperature of $T_{eff} = 25,700$K and a gravity of $\log_{10} g = 4.27$. If we hold the polar radius constant and follow eqn (54) and the discussion at the end of §3, we can plot out the trajectory of the star's surface conditions in the $T\ vs. \log g$ plane. Then, looking at our tables of $I(\mu)$ and $Q(\mu)$ for the TLUSTY models of hot stellar atmospheres, we find three temperatures, 20,000K, 23,000K and 27,500K which span the range of surface temperatures for this star rotating at $w = 0.95$. The surface gravities at 20,000, 23,000 and 27,500 are log g = 3.68, 3.93 and 4.27. We can first interpolate between models with log g = 3.5 and 4.0 or 4.0 and 4.5 to obtain models with log g of 3.68, 3.93 and 4.25 at the three temperatures. We next interpolate between these three models to obtain $I(\mu)$ and $Q(\mu)$ for each $T(\theta)$ on the star's surface, which we insert into the equations for the Stokes parameters integrated over the stellar surface:

$$I(i) \ = \ 2R_p^2 \int_0^{\pi/2} \int_0^{\pi} I(\theta, |\mu|)\ |\mu|\ \frac{x^2 \sin\theta}{\cos(\theta - \delta)}\ d\phi\ d\theta \tag{75}$$

and

$$Q(i) \ = \ 2R_p^2 \int_0^{\pi/2} \int_0^{\pi} Q(\theta, |\mu|)\ \cos(2\xi)\ |\mu|\ \frac{x^2 \sin\theta}{\cos(\theta - \delta)}\ d\phi\ d\theta \ \ . \tag{76}$$

Here, the $I(\mu)$ and $Q(\mu)$ are from the stellar atmosphere models and are in physical units; they include the variation of flux with surface temperature.

We have written programs in the 'J' language to carry out the integrations. The results are a strong function of wavelength. A sample is shown in Fig. (11). At visible wavelengths, the polarization is aligned with the rotation axis ($Q(i) > 0$), while shortward of 3000Å, it is perpendicular to it ($Q(i) < 0$).

It turns out that for this type of star, there is hardly any polarization in the visible part of the spectrum (less than 0.016% !). Not only are the values of $Q(\mu)$ small in the visible, but $Q(\mu)$ is positive for $\mu >\sim 0.2$ and negative near the limb ($\mu <\sim 0.2$). As a result there is a lot of cancellation. As we move toward the ultraviolet, Q becomes more negative, and at around 3000Å the net polarization vanishes.

The magnitude of the polarization is greater in the far ultraviolet, so that out at 1200Å, we may get half a percent or so. See Figs. (12),(13),(14) and (15). One reason for this modest amount of net polarization is that the star we have chosen is a main sequence star with a high gravity - the lower the gravity, the higher the scattering to absorption ratio. Unfortunately, giants are not fast rotators. We will explore other stellar types (e.g., spectral type K) and post the results here later.

This question may be mostly academic, since the polarization of an isolated rapid rotator, even if we should have a means to observe it in the UV, would be hard to separate from interstellar polarization.

Perhaps a more interesting subject for investigation would be the polarization from a star distorted in the form of a Roche equipotential by a companion. That is because a system viewed perpendicular to the orbital plane (which would show no eclipses) would have a net polarization that would rotate in angle with the orbital period and thus could be distinguished from the interstellar component.

5.3. Continuum Polarization Revealed by Doppler Shifted Absorption Lines

An effect of stellar rotation, first proposed in 1946 by Y. Öhman (Ap.J.,104,460), is polarization across absorption lines resulting from the drop in the continuum polarization in the line due to the additional absorption. When the rotation-induced shifts of the line profile are considered, the line will effectively block a strip parallel to the rotation axis, which scans across the star as a function of wavelength. Thus, e.g., the continuum polarization from the limb near the equator will not be canceled by the (perpendicular) polarization from the polar limb regions, since they are blocked by the line profile. *Note that this effect has nothing to do with line polarization or the Hanle effect*; it is merely that a Doppler-shifted line masks part of the star, breaking the symmetry and letting us see the continuum polarization. Detection of this polarization would reveal the orientation of the star's rotation axis. (This topic was later developed by Collins and Cranmer, 1991, M.N.R.A.S., 253, 167.)

The velocity of the stellar surface is given by

$$v(\theta) \;=\; v_0 \; w \; x(\theta, w) \; \sin\theta \;\; , \quad \text{where} \quad v_0 \;=\; 237.8 \; [M/R_p]^{1/2} \;\; km/sec \;\; , \tag{77}$$

and the stellar mass M and polar radius R_p are in solar units. The component of $v(\theta)$ along the line of sight to the surface element is given by

$$v_{obs} \;=\; v(\theta) \; \sin\phi \; \sin i \tag{78}$$

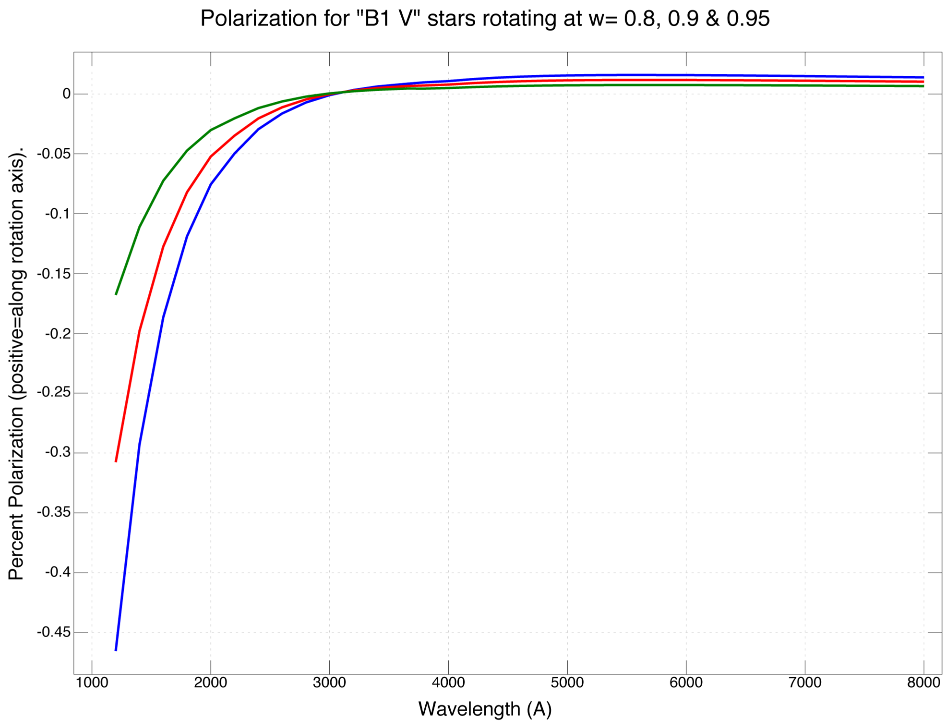

Fig. 11.— The percent net polarization vs wavelength for various rotation velocities. The angle of inclination is 90^o and the stellar spectral type is B1 V.

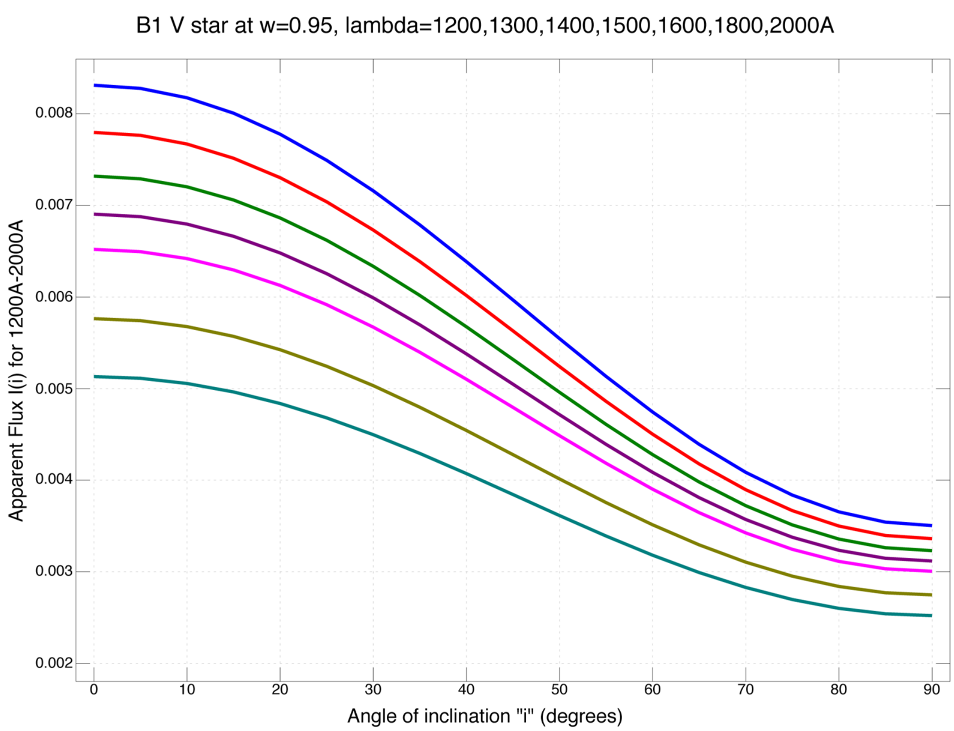

Fig. 12.— The flux vs inclination for various far UV wavelengths. The rotational velocity parameter w is 0.95 for all models. Color code: blue=1200Å, red=1300Å, green=1400Å, etc.

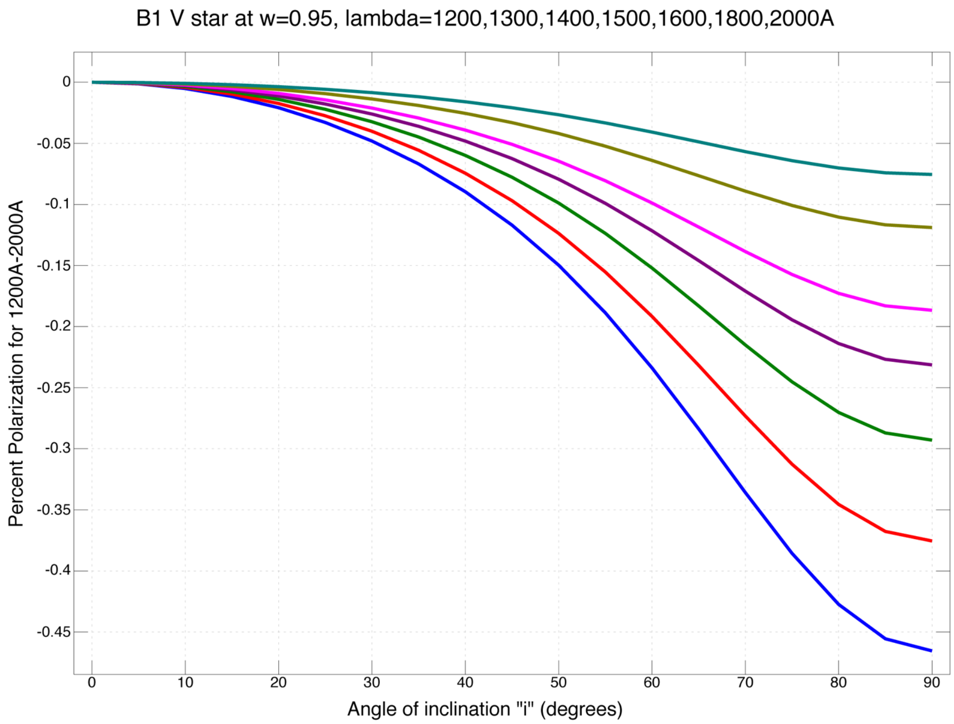

Fig. 13.— Percentage polarization 100*Q(i)/I(i) vs i for various far UV wavelengths. The rotational velocity parameter w is 0.95 for all models. Blue=1200Å, red=1300Å, green=1400Å etc.

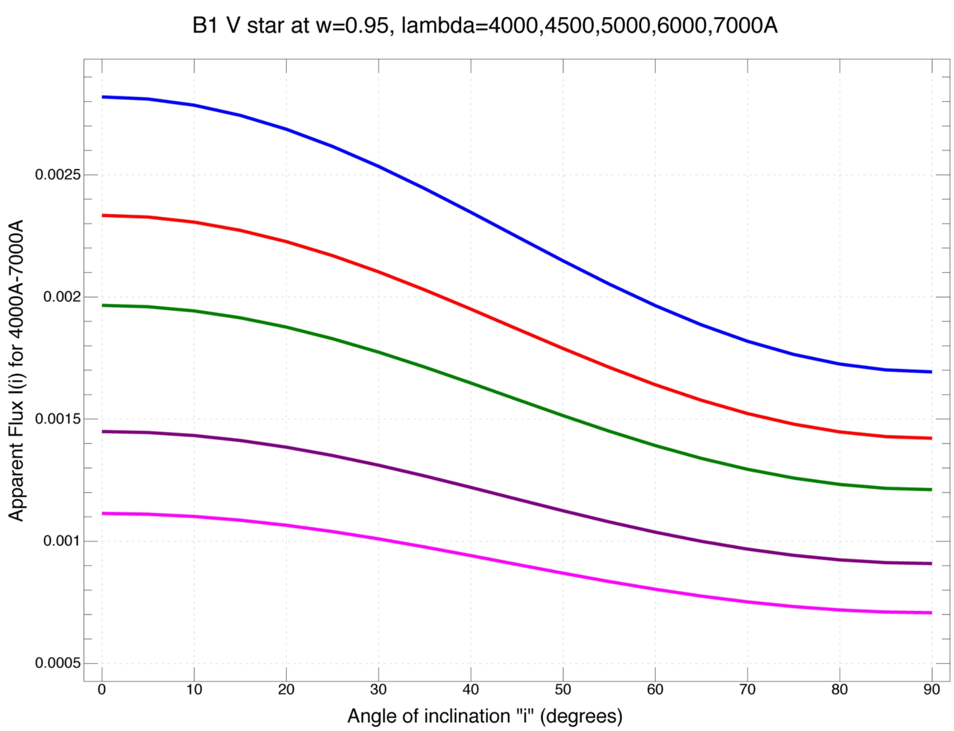

Fig. 14.— The flux vs inclination for various visible wavelengths. The rotational velocity parameter w is 0.95 for all models. Color code: blue=4000Å, red=4500Å, green=5000Å, etc.

and this velocity will produce a Doppler shift of $\Delta\lambda \; = \; \lambda \; v_{obs}/c$. Thus the Doppler shift at a given point on the stellar surface can be written

$$\Delta\lambda(\theta,\phi) \; = \; 7.932 \times 10^{-4} \; \lambda \; [M/R_p]^{1/2} \; w \; x(\theta,w) \; \sin i \; \sin\theta \; \sin\phi \quad , \tag{79}$$

where λ and $\Delta\lambda$ are in Å units. This is the shift that we apply to the Stokes parameters I and Q across the line profiles at each point on the stellar surface when integrating Eqns. (70) and (71).

To carry out the calculation, we need the emergent $I(\mu)$ and $Q(\mu)$ of the stellar atmosphere at sufficient wavelength resolution to follow the changes across the spectral lines. The Doppler width due to thermal motions in a hot star may be of the order of 0.02Å for a metal line. The simplest approach is to neglect any polarization due to line scattering and simply treat the lines as sources of pure LTE absorption. We have given some results for such a "second stellar spectrum" of hot stars:

http://www.astro.umd.edu/~jph/2ND_STELLAR_SPECTRUM/

There we computed the intensity and polarization with a wavelength resolution of $\sim$ 0.0025Å. Now the Doppler shifts produced by *rapidly* rotating stars may be of the order of 1Å. This would smear out this effect except for the strongest lines, which are likely to involve line polarization effects. But for slowly rotating stars, or stars seen at a small angle of inclination, so that the maximum rotational Doppler shift is of the order of 10-30 km/sec, we may expect such effects to appear. Using the results tabulated at the above web page reference, we have confirmed these expections.

REFERENCES

Chandrasekhar, S., 1946, Ap.J., **103**, 351.

Chandrasekhar, S., 1960, *Radiative Transfer*, p248, Dover Publications, New York.

Espinosa Lara, F. and Rieutord, M., 2011, A&A, **533**, A43.

Harrington, J. P., and Collins, G. W. II, 1968, Ap.J., **151**, 1051.

Rieutord, M. and Espinosa Lara, F., 2009, Comm. in Astroseismology, **158**.

von Zeipel, H. 1924, MNRAS, **84**, 665.

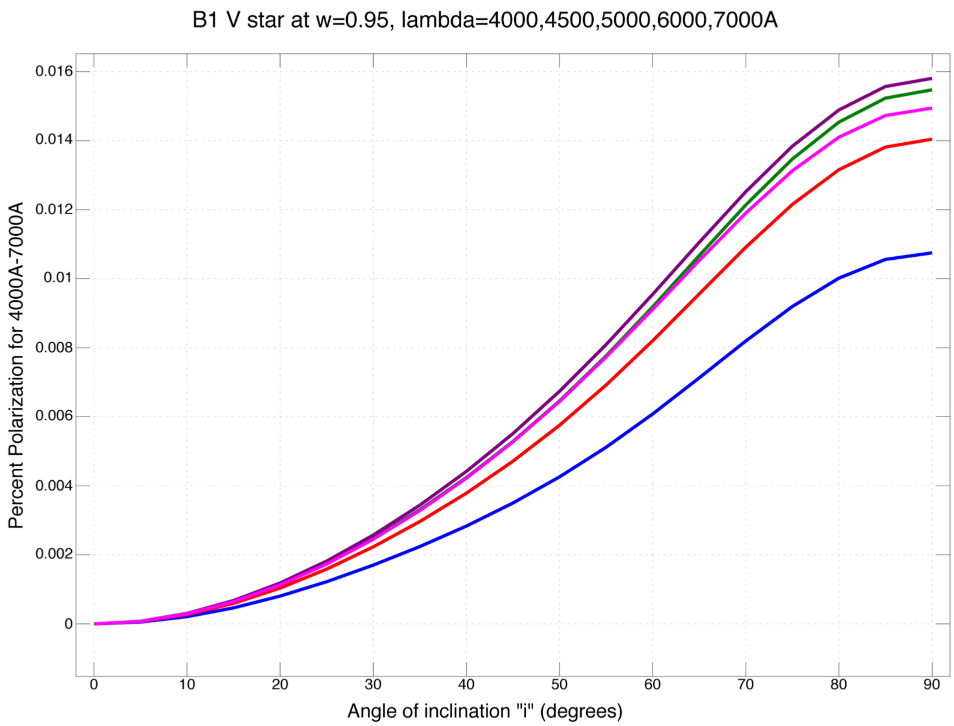

Fig. 15.— Percentage polarization 100*Q(i)/I(i) vs i for various visible wavelengths The rotational velocity parameter w is 0.95 for all models. Blue=4000Å, red=4500Å, green=5000Å etc.