Polarization source term $p(\tau)$ and $Q(\mu)$ from linear $B(\tau) = a + b\tau$

We may use the formal solutions of the transfer equations for polarized radiation resulting from dipole scattering, to write integral equations for the source terms $s_\nu(\tau_\nu)$ and $p_\nu(\tau_\nu)$ (Harrington, 1970):

$$s_\nu(\tau_\nu) \;=\; (1-\lambda_\nu)\;\left[\Lambda_{\tau_\nu}(s_\nu) + \frac{1}{3} M_{\tau_\nu}(p_\nu)\right] \;+\; \lambda_\nu B_\nu(\tau_\nu) \tag{1}$$

$$p_\nu(\tau_\nu) \;=\; \frac{3}{8}(1-\lambda_\nu)\;\left[M_{\tau_\nu}(s_\nu) + N_{\tau_\nu}(p_\nu)\right] \tag{2}$$

Here, $s(\tau)$ and $p(\tau)$ are the source terms. Λ_τ is the familiar Λ-operator,

$$\Lambda_\tau\left\{f(t)\right\} \;=\; \frac{1}{2}\int_0^\infty f(t)\; E_1(|t-\tau|)\; dt \quad . \tag{3}$$

The less familiar M_τ and N_τ operators arise when considering polarized radiation. In particular, M_τ, operating on some function $f(t)$ is defined as

$$M_\tau\left\{f(t)\right\} \;=\; \int_0^\infty f(t)\left[\frac{1}{2}E_1(|t-\tau|) - \frac{3}{2}E_3(|t-\tau|)\right]\; dt \tag{4}$$

We will make extensive use of the M_τ operator in what follows.

1. Approximate source term $p(\tau)$ for linear $B(\tau)$.

We consider a source term linear in τ:

$$B(\tau) \;=\; s(\tau) = \; a \;+\; b\tau \quad \text{where } a \text{ and } b \text{ are constants.} \tag{5}$$

We will consider the polarization to be small, so that $p(\tau) << s(\tau)$. Then if we define $\Sigma_\nu = (1-\lambda_\nu) = \sigma_\nu/(\kappa_\nu+\sigma_\nu)$ we have the approximation

$$p(\tau) = \frac{3}{8}\;\Sigma_\nu\; M_\tau(s(\tau)) \;=\; \frac{3}{8}\;\Sigma_\nu\; M_\tau(a+b\tau) \tag{6}$$

We ultimately want to evaluate the polarization of the emergent radiation, $P(0,\mu) \;=\; Q(0,\mu)/I(0,\mu)$ where $I(0,\mu)$ is the intensity of radiation emerging at $\mu = \cos(\theta)$ and θ is the angle between the ray and the normal to the surface. $Q(0,\mu)$ is the Stokes Q-parameter. I and Q can be obtained from integrals of s and p:

$$I_\nu(0,\mu) \;=\; \int_0^\infty \left\{s_\nu(\tau_\nu) + \left(\frac{1}{3}-\mu^2\right)p_\nu(\tau_\nu)\right\}\; e^{-\tau_\nu/\mu}\;\frac{d\tau_\nu}{\mu} \tag{7}$$

$$Q_\nu(0,\mu) \;=\; \int_0^\infty \;\left\{\left(1-\mu^2\right)p_\nu(\tau_\nu)\right\}\; e^{-\tau_\nu/\mu}\;\frac{d\tau_\nu}{\mu} \tag{8}$$

In equation (7), we neglect the term involving $p(\tau)$ and evaluate the remaining term:

$$I_\nu(0,\mu) \;=\; \int_0^\infty \; s_\nu(\tau_\nu)\; e^{-\tau_\nu/\mu}\;\frac{d\tau_\nu}{\mu} \;=\; \int_0^\infty \;(a+b\tau)\; e^{-\tau_\nu/\mu}\;\frac{d\tau_\nu}{\mu} \;=\; a \;+\; b\;\mu \tag{9}$$

This is the simple limb darkening law that follows from a linear Plank function. The evaluation of $Q(0, \mu)$ is not so simple. We first must evaluate the M_τ transform of $s(\tau)$:

$$M_\tau(s) \;=\; M_\tau(a \;+\; b\;\tau) \;=\; a\; M_\tau(1) \;+\; b\; M_\tau(\tau) \tag{10}$$

The first term is straitforward; the M-transform of unity is

$$M_\tau(1) \;=\; \frac{3}{2}\; E_4(\tau) \;-\; \frac{1}{2}\; E_2(\tau) \tag{11}$$

The transform of the 1st moment τ is more complex:

$$M_\tau(\tau) \;= \frac{9}{2}\; E_5(\tau) \;-\; \frac{1}{2}\; E_3(\tau) \;-\; e^{-\tau} \;+\; \tau\; M_\tau(1) \tag{12}$$

We see that $M_\tau(1)$ reappears, multiplied by τ. To simplify this further, we use the recurrence relations for exponential integrals:

$$\tau\; E_2(\tau) \;=\; e^{-\tau} \;-\; 2\; E_3(\tau) \tag{13}$$

$$\tau\; E_4(\tau) \;=\; e^{-\tau} \;-\; 4\; E_5(\tau) \tag{14}$$

Inserting these expressions into equation (12) we have

$$M_\tau(\tau) \;=\; \frac{9}{2}\; E_5(\tau) \;-\; \frac{1}{2}\; E_3(\tau) \;- e^{-\tau} \;+\; \frac{3}{2}\; [e^{-\tau} - 4\; E_5(\tau)] \;-\; \frac{1}{2}\; [e^{-\tau} - 2\; E_3(\tau)] \tag{15}$$

The $e^{-\tau}$'s cancel out leaving us with

$$M_\tau(\tau) \;=\; \frac{1}{2}\; E_3(\tau) \;-\; \frac{3}{2}\; E_5(\tau) \tag{16}$$

Thus our formula for $p(\tau)$ is

$$p(\tau) = \frac{3}{16}\; \Sigma_\nu\; \{a\;\; (3\; E_4(\tau) - E_2(\tau)) \;+\; b\;\; (E_3(\tau) \;-\; 3\; E_5(\tau))\} \tag{17}$$

We have evaluated $p(\tau)$ for $a = 1$ and various values of b as shown in Fig. 1. (We set $\Sigma = 0.1$ and assumed it was constant with τ.) The case with $b = 0$ (the top blue curve) is an isothermal atmosphere. Note that for this case, p is everywhere positive, which means that the polarization is along the z-axis (along the normal to the surface). The situation changes when we set a positive slope b to the Planck function. Then p goes negative, increasingly so as the slope of B increases. Note that the $a = 1,\; b = 1.5$ curve is Eddington's elementary approximation to the grey atmosphere.

1.1. The integration over τ to obtain $Q(0,\mu)$

We have given $I_\nu(0,\mu)$ in equation (9). We now turn to $Q(0,\mu)$ as defined in equation (8). Inserting equation (17) we have

$$Q_\nu(0,\mu) \;=\; \frac{3}{16}\;\left(1-\mu^2\right)\;\int_0^\infty \;\Sigma_\nu\;\left\{a\,(3\;E_4(\tau)-E_2(\tau))\;+\;b\,(E_3(\tau)-3\;E_5(\tau))\right\}\;e^{-\tau_\nu/\mu}\;\frac{d\tau_\nu}{\mu} \qquad (18)$$

We see that we must evaluate the integral of $e^{-\tau/\mu}$ times various exponential integrals. These can be written as Laplace transforms of the exponential integrals (Kourganoff, 1952, Appendix I, eqn 37.1):

$$\mathcal{L}_a(E_n(t)) \;=\; a\int_0^\infty \;e^{-at}E_n(t)dt \qquad (19)$$

The Laplace transforms are relatively simple. With $a = 1/\mu$ we find that:

$$\mathcal{L}_{1/\mu}(E_2(t)) \;=\; 1\;-\mu\;\log[1+1/\mu] \qquad (20)$$

$$\mathcal{L}_{1/\mu}(E_3(t)) \;=\; \frac{1}{2}\;-\mu\;+\mu^2\log[1+1/\mu] \qquad (21)$$

$$\mathcal{L}_{1/\mu}(E_4(t)) \;=\; \frac{1}{3}\;-\;\frac{1}{2}\mu\;+\mu^2\;-\;\mu^3\;\log[1+1/\mu] \qquad (22)$$

$$\mathcal{L}_{1/\mu}(E_5(t)) \;=\; \frac{1}{4}\;-\;\frac{1}{3}\mu\;+\frac{1}{2}\mu^2\;-\;\mu^3\;+\;\mu^4\;\log[1+1/\mu] \qquad (23)$$

so that we have

$$\int_0^\infty M_\tau(1)e^{-\tau/\mu}\frac{d\tau}{\mu} \;=\; -\frac{3}{4}\;\mu+\frac{3}{2}\;\mu^2+\frac{1}{2}\;\mu(1-3\mu^2)\;\log[1+1/\mu] \qquad (24)$$

and

$$\int_0^\infty M_\tau(t)e^{-\tau/\mu}\frac{d\tau}{\mu} \;=\; \frac{1}{4}\;-\;\frac{3}{4}\mu^2\;+\;\frac{3}{2}\mu^3\;+\;\frac{1}{2}\;\mu^2(1-3\mu^2)\;\log[1+1/\mu] \qquad (25)$$

Thus or final expression for the Stokes Q-parameter is

$$Q_\nu(0,\mu) \;=\; \frac{3}{16}\;\left(1-\mu^2\right)\;\Sigma_\nu\;\Bigg\{a\left(-\frac{3}{2}\;\mu+\;3\;\mu^2+\;\mu(1-3\mu^2)\;\log[1+1/\mu]\right)\;+$$
$$b\left(-\;\frac{1}{4}\;-\;\frac{3}{2}\mu^2\;+\;3\mu^3\;+\;\mu^2(1-3\mu^2)\;\log[1+1/\mu]\right)\Bigg\} \qquad (26)$$

And it doesn't even have an exponential integral!

Note that this expression assumes that Σ is constant with τ. It is possible to include a variable scattering with depth, and we give an example in the next secion. In this sense our model is not realistic, as in any real hot star atmosphere, Σ will surely increase as $\tau \to 0$.

We have plotted the percent polarization, $-100 * Q(\mu)/I(\mu)$, obtained from this equation and $I(\mu)$ from equatiion (9). (We plot -1 times the polarization for convience.) We take $a = 1$ and $b = 0, 0.5, 1.0, 1.5, 2.0$ and 2.5, with $\Sigma = 0.1$. We see the steep rise in polarization with increasing slope b (negative, angle perpendicular to the z-axis). We also see small, but entirely positive polarization for $b = 0$ (isothermal), and for a slope less than the $b = 1.5$ grey body case, such as $b = 0.5$, we see positive values near the disk center switching to negative values towards the limb. This is the behavior that is seen for hot stars at visual wavelengths.

1.2. A case where Σ_ν is a function of τ_ν.

In hot stellar atmospheres, where the scattering is due mostly to free electrons,the fraction of the opacity due to scattering will typically rise as we approach the surface. This is because the density decreases exponentially with some scale height, and the Saha equation leads to higher ionization with a corresponding decrease in continuous absorption. We can investigate this sort of behavior in our analytic model by considering a variable Σ_ν. Let us set

$$\Sigma_\nu(\tau_\nu) \;=\; \Sigma_0 \; e^{-s\tau_\nu} \;, \tag{27}$$

where s is a free parameter. This is a function which has the value Σ_0 near the surface ($\tau \;\to\; 0$) but decreases expontentially as we go into the atmosphere, with s determining the rapidity of the decrease. In Fig. 3 we show the behavior of $\Sigma(\tau)$ as a function of $\log(\tau)$ for $\Sigma_0 = 1$ and $s = 0.1, 0.5, 1, 2, 5, 10, 20, 50$.

Returning to equation (18), we see we must integrate

$$Q_\nu(0,\mu) \;=\; \frac{3}{16}\left(1-\mu^2\right)\Sigma_0 \int_0^\infty \;\; \{a\,(3\;E_4(\tau) - E_2(\tau)) \;+\; b\,(E_3(\tau) - 3\;E_5(\tau))\}\; e^{-s\tau}\; e^{-\tau/\mu}\;\frac{d\tau}{\mu} \tag{28}$$

We see that if we define α as

$$\alpha \;=\; s \;+\; \frac{1}{\mu} \;=\; \frac{1+s\;\mu}{\mu} \quad \text{and} \quad \beta \;=\; \frac{1}{\alpha} \;=\; \frac{\mu}{1+s\;\mu} \tag{29}$$

then the equation has the same form as equation (18):

$$Q_\nu(0,\mu) \;=\; \frac{3}{16}\left(1-\mu^2\right)\frac{\Sigma_0}{\mu\alpha}\;\left\{\alpha\int_0^\infty \;[a\,(3\;E_4(\tau) - E_2(\tau)) \;+\; b\,(E_3(\tau) - 3\;E_5(\tau))]\;\; e^{-\alpha\;\tau}\; d\tau\right\} \tag{30}$$

We see that the integral in braces is just the same series of Laplace transforms as defined by equations (19)-(23). Our expression for $Q_\nu(0,\mu)$, with μ replaced by β from equation (29), becomes

$$\begin{aligned} Q_\nu(0,\mu) \;=\; \frac{3}{16}\;\left(1-\mu^2\right)\;\frac{\Sigma_0}{1+s\;\mu}\;&\left\{a\left(-\frac{3}{2}\,\beta +\; 3\;\beta^2 +\; \beta(1-3\beta^2)\;\log[1+1/\beta]\right) \;+\right. \\ &\left. b\left(-\;\frac{1}{4}\;-\;\frac{3}{2}\beta^2 \;+\; 3\beta^3 \;+\; \beta^2(1-3\beta^2)\;\log[1+1/\beta]\right)\right\} \end{aligned} \tag{31}$$

We see that this expression reduces to the earlier equation (26) when $s = 0$. In Fig. 4 we show some examples with $a = 1, b = 1.5, \Sigma_0 = 0.1$ and values $s \;=\; 0, 0.1, 0.5, 1, 2, 5, 10, 20$ & 50. We see that as s increases, the polarization is depressed. Furthermore, if you look at the *fractional decrease* in the polarization for a given s, it is greatest near $\mu = 1$, becoming less as we approach $\mu = 0$, where all curves converge to the same limb value. This is because as $\mu \to 0$, we only see the vary surface layers, where $\Sigma(\tau) \to \Sigma_0$.

J. Patrick Harrington 15 October 2025

REFERENCES

Harrington, J. P., 1970, Ap. and Space Science, **8**, 227.

Kourganoff, V., 1952, "Basic Methods in Transfer Problems", Oxford Clarendon Press.