Problem Set No. 4 – solutions.

① (a) The Lane-Emden equation is $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$

To show that $\theta(\xi) = \sin\xi/\xi$ is a solution, we evaluate

$$\frac{d\theta}{d\xi} = \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}, \text{ so } \left(\xi^2\frac{d\theta}{d\xi}\right) = \xi\cos\xi - \sin\xi$$

Next, $\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \cos\xi - \xi\sin\xi - \cos\xi = -\xi\sin\xi$

So the l.h.s. is just $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\frac{\sin\xi}{\xi}$

But the r.h.s. is $-\theta^n = -\theta^1 = -\frac{\sin\xi}{\xi}$, so the L-E equation is satisfied.

Use the expansion $\sin\xi = \xi - \frac{\xi^3}{3!} + \frac{\xi^5}{5!} - \frac{\xi^7}{7!} + \dots$

to obtain $\theta(\xi) = \frac{\sin\xi}{\xi} = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} - \frac{\xi^6}{5040} + \dots$

Clearly, $\theta(\xi=0) = 1$ and $\frac{d\theta}{d\xi} = -\frac{1}{3}\xi + \frac{1}{30}\xi^3 - \frac{1}{840}\xi^5 + \dots$

So $d\theta/d\xi = 0$ at $\xi = 0$.

(b) We gave the expansion above. For $n=1$ we have

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{1}{120}\xi^4 - \frac{1(8-5)}{15120}\xi^6 + \dots \quad \text{but } \frac{3}{15120} = \frac{1}{5040}$$

so the series are identical.

(c) $$\left(d\theta/d\xi\right)_{\text{at }\xi=\pi} = \left[\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right] = \left[\frac{-1}{\pi} - \frac{0}{\pi^2}\right] = \underline{\underline{-\frac{1}{\pi}}}$$

(d) $$D_n = -\left[\frac{3}{\pi}\left(-\frac{1}{\pi}\right)\right]^{-1} = \left(\frac{3}{\pi^2}\right)^{-1} = \frac{\pi^2}{3} = \underline{\underline{3.28987}}$$

② (a) Combining the unnumbered equation between (1.5) and (1.6) with equation (1.6) we have

$$E_{GR} = \int_0^R 4\pi r^3 \frac{dP}{dr}\,dr$$

We are considering the model where $\frac{dP}{dr} = -\frac{4\pi}{3} G \rho_c^2 r e^{-(r/a)^2}$.

Thus we have $E_{GR} = -\int_0^R 4\pi r^3 \frac{4\pi}{3} G\rho_c^2 r e^{-(r/a)^2} dr = -3\left(\frac{4\pi}{3}\right)^2 \rho_c^2 G \int_0^R r^4 e^{-(r/a)^2} dr$

Let $x = r/a$ so $r = ax$ & $dr = a\,dx$. Then

$$E_{GR} = -3\left(\frac{4\pi}{3}\right)^2 \rho_c^2 G a^5 \int_0^{R/a} x^4 e^{-x^2} dx$$

But if $a \ll R$, $R/a \gg 1$ (e.g. if $a = \frac{1}{5}R$, $\frac{R}{a} = 5$ so $e^{-x^2} \to e^{-25}$) the value of the integral is nearly unchanged if we let the upper limit $\to \infty$. Then $\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$ and we find

$$E_{GR} = -3\left(\frac{4\pi}{3}\right)^2 \rho_c^2 G a^5 \frac{3\sqrt{\pi}}{8}.$$ Now $M = \left(\frac{4\pi}{3}\right)\rho_c \frac{a^3\sqrt{6}}{3}$

(eq. 5.32) so that $\left(\frac{4\pi}{3}\right)^2 \rho_c^2 = \frac{1}{6a^6} M^2$. Combine the two and

$$E_{GR} = -3G\frac{a^5}{6a^6} M^2 \frac{3\sqrt{\pi}}{8} = -\frac{3\sqrt{\pi}}{16}\frac{1}{a} GM^2$$

to put this into the familiar GM^2/R form for the potential, we write $\frac{1}{a} = \left(\frac{R}{a}\right)\frac{1}{R}$ to obtain

$$E_{GR} = -\frac{3\sqrt{\pi}}{16}\left(\frac{R}{a}\right)\frac{GM^2}{R} = -0.3323\left(\frac{R}{a}\right)\frac{GM^2}{R} \simeq -\frac{1}{3}\left(\frac{R}{a}\right)\frac{GM^2}{R}$$

(b) If $(R/a) = 5$, then $E_{GR} = -\frac{5}{3}\frac{GM^2}{R} = -f\frac{GM^2}{R}$ with $\underline{\underline{f = \frac{5}{3}}}$

But for a polytrope of index n, $f = \frac{3}{5-n}$ (class notes)

Thus $n = 5 - \frac{3}{f}$ so $n = \frac{16}{5} = \underline{\underline{3.2}}$

(3) We are considering the gas and radiation at the center of a star. β is P_g / P_c, where P_c is the total pressure. Then $(1-\beta) = P_r / P_c$. The gas pressure is just

$$P_g = nkT_c = \frac{\rho_c}{\bar{m}} kT_c \quad \text{so} \quad \beta = \frac{k}{\bar{m}} \rho_c \frac{T_c}{P_c}$$

The radiation pressure is $P_r = \frac{1}{3} a T_c^4$ where a is the radiation constant $a = 7.566 \cdot 10^{-15}$ erg $cm^{-3} K^{-4}$. Thus we have

$$(1-\beta) = \frac{a}{3} \frac{T_c^4}{P_c}$$ Following p. 237 of Phillips, we

evaluate $$\frac{(1-\beta)}{\beta^4} = \left(\frac{a}{3} \frac{T_c^4}{P_c}\right)\left[\left(\frac{k}{\bar{m}}\right)^4 \rho_c^4 \frac{T_c^4}{P_c^4}\right]^{-1} = \frac{a}{3}\left(\frac{\bar{m}}{k}\right)^4 \frac{P_c^3}{\rho_c^4}$$

Now we have $P_c < \left(\frac{\pi}{6}\right)^{1/3} G M^{2/3} \rho_c^{4/3}$ so $P_c^3 < \frac{\pi}{6} G^3 M^2 \rho_c^4$

Put this into the previous expression to get

$$\frac{(1-\beta)}{\beta^4} < \frac{\pi}{18} a \left(\frac{\bar{m}}{k}\right)^4 G^3 M^2$$

Let's write $\bar{m}$ as μm_H, where μ is the mean molecular weight in units of the hydrogen mass. Then

$$\frac{1-\beta}{\beta^4} < C_1 \mu^4 \left(\frac{M}{M_\odot}\right)^2 \quad \text{where } C_1 = \frac{\pi}{18} a \left(\frac{m_H}{k}\right)^4 G^3 M_\odot^2$$

Putting in the constants, we have $C_1 = 0.033515$

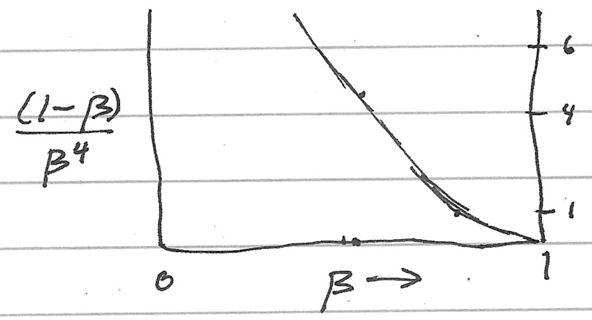
For a mix of hydrogen & helium $\mu \triangleq 4/(3+5X)$ so for the original solar composition, $X = 0.7$, $Y = 0.3$, we get $\mu \cong 0.6$

So finally $$\frac{1-\beta}{\beta^4} < 0.00434 \left(\frac{M}{M_\odot}\right)^2$$

For $M = 4 M_\odot$, $$\frac{1-\beta}{\beta^4} < 0.0695$$

and for $40 M_\odot$ $$\frac{1-\beta}{\beta^4} < 6.95$$

③ cont. Now, if we were to plot $(1-\beta)/\beta^4$ over the range of $0 \leq \beta \leq 1$, the curve would look like this:



So the higher β, the smaller $(1-\beta)/\beta^4$.

That means the inequality will be reversed for β.

By plugging some values we find that $(1-\beta)/\beta^4 = 0.0659$ when $\beta = 0.944655$ and also $(1-\beta)/\beta^4 = 6.95$ for $\beta = 0.514187$. Thus our resulting limits are

$$M = 4M_\odot, \quad \beta > 0.944655 \Rightarrow \frac{P_{rad}}{P_c} = (1-\beta) < 0.055$$

$$M = 40\,M_\odot, \quad \beta > 0.514187 \Rightarrow P_{rad}/P_c = (1-\beta) < 0.4858$$

Radiation can't have more than a 5% contribution at the center of a $4M_\odot$ star, but can have up to 50% of the pressure for a $40\,M_\odot$ star.

4 (Phillips 6.1) The density of atomic matter is of the order of

$$\rho_{at} = \frac{m_H}{a_B^3} = m_H \alpha_{EM}^3 \frac{m_e^3 c^3}{\hbar^3} = m_H \alpha_{EM}^3 \frac{m_e^3 c^3}{h^3} (2\pi)^3$$

(this would work better if the $\hbar$ were h & we didn't get the $(2\pi)^3$!)

From equation 6.4, the central density of a (non-relativistic) degenerate object is given by

$$\rho_c = \frac{3.1}{Y_e^5} \left(\frac{M}{m_H} \alpha_G^{3/2} \right)^2 m_H \frac{m_e^3 c^3}{h^3}$$

If we set $\rho_{at} = \rho_c$ we get

$$\frac{3.1}{Y_e^5} \frac{M^2}{m_H^2} \alpha_G^3 \cancel{m_H} \cancel{\frac{m_e^3 c^3}{h^3}} = \cancel{m_H} \alpha_{EM}^3 \cancel{\frac{m_e^3 c^3}{h^3}} (2\pi)^3$$

$$M^2 = m_H^2 \frac{Y_e^5}{3.1} \alpha_G^{-3} \alpha_{em}^3 (2\pi)^3$$

$$M = \frac{Y_e^{5/2} (2\pi)^{3/2}}{(3.1)^{1/2}} \left(\frac{\alpha_{EM}}{\alpha_G} \right)^{3/2} \cdot m_H$$

We would get the result in Phillips if $\left[\frac{Y_e^5 \cdot 2\pi^3}{3.1} \right]^{1/2} \simeq 1$.

What's Y_e? The number of electrons per nucleon. Phillips gives the expression $Y_e = (1+X)/2$. For a white dwarf it would be $\frac{1}{2}$, but we are talking about big planets. For a solar composition, $X \approx 0.7$ so $Y_e \approx 0.85$. Then

$$\left[\frac{0.85^5 \cdot 2\pi^3}{3.1} \right]^{1/2} = 5.96$$

so that $M \cong 6 \left(\frac{\alpha_{EM}}{\alpha_G} \right)^{3/2} m_H$

$$\left(\frac{\alpha_{EM}}{\alpha_G} \right)^{3/2} m_H = \alpha_{EM}^{3/2} M_* = \left(\frac{1}{137} \right) 1.85\, M_\odot = 0.00115\, M_\odot$$

So I get $M_p \approx 0.007\, M_\odot = 7.3\, M_{Jupiter}$.

As I said above, if $\hbar$ was ~~h~~ h, we wouldn't have that $(2\pi)^{3/2}$ ~~and~~ and we would be closer to Phillips result.