Problem Set No. 3 - solutions

① The Rosseland mean opacity κ is defined as

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$$\frac{1}{\kappa} = \left(\frac{ac}{\pi}T^3\right)^{-1} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} d\nu$$

where B_ν is the Planck function $B_\nu = \frac{2h\nu^3}{c^2}\left(e^{h\nu/kT} - 1\right)^{-1}$.

Taking the derivative w.r.t. T, we find

$$\frac{\partial B_\nu}{\partial T} = \frac{2h\nu^3}{c^2} \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT} - 1\right)^2} \frac{h\nu}{kT} \frac{1}{T}$$

Now if $\kappa_\nu = \text{const}\, \rho\, \nu^{-3} T^{-1/2}$, then we have

$$\frac{1}{\kappa} = \left(\frac{ac}{\pi}T^3\right)^{-1} \int_0^\infty \frac{\nu^3 T^{1/2}}{\text{const}\cdot\rho} \frac{2h\nu^3}{c^2} \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT} - 1\right)^2} \frac{h\nu}{kT} \frac{1}{T} d\nu$$

now, if we define $x = \frac{h\nu}{kT}$, so $\nu = \frac{kT}{h}x$ & $d\nu = \frac{kT}{h}dx$

$$\frac{1}{\kappa} = \left(\frac{ac}{\pi}T^3\right)^{-1} \frac{1}{\text{const}\cdot\rho} \int_0^\infty T^{1/2} \left(\frac{kT}{h}\right)^3 x^3 \cdot \frac{2h}{c^2}\left(\frac{kT}{h}\right)^3 x^3 \frac{e^x}{(e^x-1)^2} x \frac{1}{T}\left(\frac{kT}{h}\right) dx$$

$$\frac{1}{\kappa} = C_1 \frac{1}{\rho} \frac{T^{1/2}}{T^3} T^3 T^3 \frac{1}{T} T \int_0^\infty \frac{x^7 e^x}{(e^x-1)^2} dx$$

where we have folded all the constants into C_1.

Thus $$\frac{1}{\kappa} = \frac{C_1}{\rho} T^{7/2} \int_0^\infty \frac{x^7 e^x}{(e^x-1)^2} dx$$

Now $\int_0^\infty \frac{x^7 e^x}{(e^x-1)^2} dx = I^*$, just some constant value.

So $$\frac{1}{\kappa} = \frac{C_1}{\rho} T^{3.5} I^* \longrightarrow \kappa \propto \rho T^{-3.5}$$

which is Kramer's law, which we wanted to establish.

(2) Let's define $\tau \equiv \dfrac{T_I}{70\cdot 10^6 K}$. Then equation 3.45 is

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$$\tau^{-3.5}\, d\tau = -\frac{\alpha}{70\cdot 10^6}\, dt$$

integrating this equation gives us

$$\frac{1}{2.5}\,\tau^{-2.5} = \frac{\alpha}{70\cdot 10^6}\, t + C_1'$$

what is the constant of integration C_1'? Suppose at $t=0$ the temperature T_I is very high: $T_I \to \infty$. Then $\tau \to \infty$ and $\tau^{-2.5} \to 0$. Thus $C_1' = 0$, if we measure the time t from the point of very high T_I.

Set $C_1' = 0$ and invert the equation:

$$\frac{5}{2}\,\tau^{2.5} = \frac{70\cdot 10^6}{\alpha}\,\frac{1}{t}$$

Now $\frac{5}{2}\tau^{2.5} = \frac{5}{2}\frac{1}{\tau}\,\tau^{3.5}$ but from equation 3.43

$$\frac{L}{L_\odot} = \left[\frac{T_I}{70\cdot 10^6}\right]^{3.5}\frac{M}{M_\odot} \Rightarrow \tau^{3.5} = \frac{L}{L_\odot}\frac{M_\odot}{M} \quad \text{So we get}$$

$$\frac{5}{2}\frac{1}{\tau}\frac{L}{L_\odot}\frac{M_\odot}{M} = \frac{70\cdot 10^6}{\alpha}\frac{1}{t} \quad \text{Invert this to get}$$

$$t = \frac{2}{5}\frac{L_\odot}{L}\frac{M}{M_\odot}\left(70\cdot 10^6\,\tau\right)\frac{1}{\alpha} = \frac{2}{5}\frac{L_\odot}{L}\frac{M}{M_\odot}\,T_I\left(\frac{3k}{2}\frac{M_\odot}{12 m_H L_\odot}\right)$$

So finally, the desired result:

$$t = \frac{3}{5}\frac{kT_I}{L}\frac{M}{12 m_H}$$

(3) We want to find the value of E for which the function

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$$f = e^{-E/kT} e^{-(E_G/E)^{1/2}} = \exp\left[-\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right]$$

is a maximum. We thus set $df/dE = 0$

$$\frac{df}{dE} = f \frac{d}{dE}\left[-\frac{E}{kT} - \left(\frac{E_G}{E}\right)^{1/2}\right] = f\left[-\frac{1}{kT} + \frac{1}{2}\frac{E_G^{1/2}}{E^{3/2}}\right] = 0$$

Thus $\frac{2}{kT} = \frac{E_G^{1/2}}{E^{3/2}} \Rightarrow E^{3/2} = E_G^{1/2} \frac{kT}{2}$

So the energy of the maximum is $\underline{\underline{E_0 = E_G^{1/3} \left(\frac{kT}{2}\right)^{2/3}}}$

8 Evaluating the form $E_0 = 1.22\left(Z_i^2 Z_j^2 \mu T_6^2\right)^{1/3}$ keV

for (a) the proton-proton chain at $T_6 = 15$

The masses are both 1, so $\mu = \frac{m_i m_j}{m_i + m_j} = \frac{1}{1+1} = \frac{1}{2}$

and $Z_i = Z_j = 1$. $E_0 = 1.22\left(\frac{1}{2} 15^2\right)^{1/3} = \underline{\underline{5.89 \text{ keV}}}$

(b) For the CNO cycle 1st step: $p + {}^{12}C \rightarrow \ldots$

$\mu = \frac{1 \cdot 12}{1+12} = \frac{12}{13}$ $Z_i = 1$ $Z_j = 6$

$$E_0 = 1.22\left(\frac{12}{13} \cdot 6^2 \cdot 15^2\right)^{1/3} = \underline{\underline{23.86 \text{ keV}}}$$

④ The solar luminosity is $L_\odot = 3.85\cdot10^{33}$ ergs/s, but we will use the rounded-off value of $L_\odot \approx 4\cdot10^{26}\,W = 4.\,10^{33}$ ergs/s as given in problem 4.3. For branch I we have the reaction $p+p \to d + e^+ + \nu_e$, and it must occur twice to produce the two ^3He nuclei for the last step. The energy released is $Q_I = 26.2$ MeV $= 26.2\cdot10^6\,eV\,(1.602\cdot10^{-12}$ erg/eV$)$ or $Q_I = 4.198\cdot10^{-5}$ erg. The number of branch I cycles to produce the sun's luminosity is $L_\odot/Q_I$ and we get 2 neutrinos/cycle so

$$\text{branch I} \quad N_\nu = 2\frac{L_\odot}{Q_I} = 2\,\frac{4\cdot10^{33}}{4.198\cdot10^{-5}} = \underline{\underline{1.906\cdot10^{38}}}\ \nu/\text{sec}$$

For branch II we have only one $p+p \to d+e^+ + \nu_e$ reaction. The reaction $e^- + {}^7Be \to {}^7Li + \nu_e$ gives another neutrino, but the problem asks only about the $p+p$ neutrinos. Thus, with $Q_{II} = 25.2$ MeV $= 4.037\cdot10^{-5}$ ergs,

$$\text{branch II} \quad N_\nu = \frac{L_\odot}{Q_{II}} = \underline{\underline{9.907\cdot10^{37}}}\ \text{neutrinos/sec}$$

The Bachall model, 85% branch I, 15% branch II, gives

$$N = 0.85\,(1.906\cdot10^{38}) + 0.15\,(9.907\cdot10^{37}) = \underline{\underline{1.77\cdot10^{38}}}\ \nu/s$$

The flux at the earth is the number spread over a sphere centered on the earth orbit with a ~~radi~~ radius $d_\oplus = 1\,AU = 1.496\cdot10^{13}$ cm.

Thus

$$\text{branch I}: \quad F_\nu = \frac{N_\nu}{4\pi d_\oplus^2} = \frac{1.906\cdot10^{33}}{4\pi(1.496\cdot10^{13})^2} = \underline{\underline{6.78\cdot10^{10}}}\ \nu/\text{cm}^2/\text{s}$$

$$\text{branch II}: \quad F_\nu = \frac{9.907\cdot10^{37}}{4\pi d_\oplus^2} = \underline{\underline{3.52\cdot10^{10}}}\ \text{neutrinos/cm}^2/\text{s}$$

& Bachall model $\to$ $\underline{\underline{6.29\cdot10^{10}}}$ neutrinos/cm²/s