Problem Set No. 2

① Look up the series expansion of arcsinh (x):

$$3\cdot\sinh^{-1}(x) = 3\left[x-\frac{1}{6}x^3+\frac{3}{40}x^5-\frac{5}{112}x^7+\dots\right]$$
$$= 3x-\frac{1}{2}x^3+\frac{9}{40}x^5-\dots$$

Also, expand $(1+x^2)^{1/2}$. We will need to keep 3 terms:

$$(1+x^2)^{1/2} = 1+\frac{1}{2}x^2-\frac{1}{8}x^4+\dots$$

So we have

$$x(2x^2-3)(1+x^2)^{1/2} = -3x+\left(2-\frac{3}{2}\right)x^3+\left(1+\frac{3}{8}\right)x^5-\dots$$

Which gives us

$$f(x) \cong \left(-3x+\frac{1}{2}x^3+\frac{11}{8}x^5\right)+\left(3x-\frac{1}{2}x^3+\frac{9}{40}x^5\right)$$
$$f(x) \cong \frac{11}{8}x^5+\frac{9}{40}x^5 = \frac{8}{5}x^5$$

Now $x \propto n^{1/3}$ and $\rho \propto n$. So $x \propto \rho^{1/3}$ and thus $x^5 \propto \rho^{5/3}$. So clearly. $P \propto f(x)$ implies $P \propto \rho^{5/3}$. Q.E.D.

(2) (a) From $y^2 + Fy - F = 0$, the quadratic formula gives us

$$y = \frac{-F \pm (F^2 - 4(-F))^{1/2}}{2} = \frac{1}{2}\left[-F \pm (F^2 + 4F)^{1/2}\right]$$

Since y must be positive (and $0 \leq y \leq 1$), we must take the positive sign. Taking the F^2 out of the root gives us

$$y = \frac{F}{2}\left[\left(1 + \frac{4}{F}\right)^{1/2} - 1\right]$$

If $F \gg 1$, $\frac{4}{F} \ll 1$ and we expand the root as

$$\left(1 + \frac{4}{F}\right)^{1/2} \cong 1 + \frac{1}{2}\left(\frac{4}{F}\right) - \frac{1}{8}\left(\frac{4}{F}\right)^2 \cong 1 + \frac{2}{F} - \frac{2}{F^2}$$

Thus $y \cong \frac{F}{2}\left[1 + \cancel{\frac{2}{F}} - \frac{2}{F^2} - 1\right] = 1 - \frac{F}{2}\frac{2}{F^2}$ so

for $F \gg 1$, $y \cong 1 - \frac{1}{F}$ & the neutral fraction is $(1-y) \cong 1/F$

Now for $F \ll 1$, $\frac{4}{F} \gg 1$ and $\left(1 + \frac{4}{F}\right)^{1/2} \cong \frac{2}{\sqrt{F}}$

also $\frac{2}{\sqrt{F}} \gg 1$ so $y \cong \frac{F}{2}\frac{2}{\sqrt{F}} \cong \sqrt{F}$

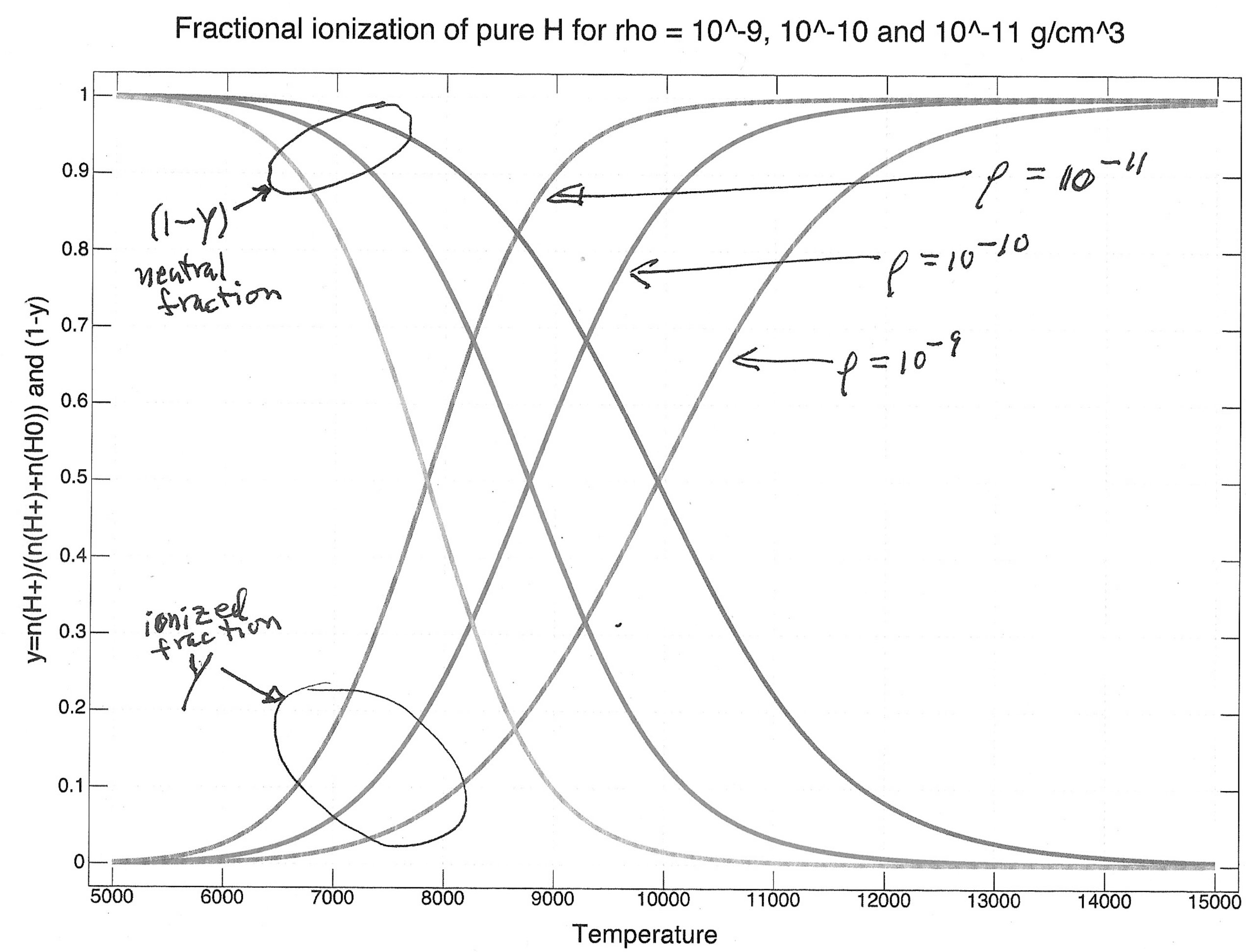
(b) We can evaluate $F = \frac{4.01 \cdot 10^{-9}}{\rho} T^{3/2} e^{-\frac{157800}{T}}$

for a particular ρ (10^{-9}, 10^{-10} or 10^{-11}) and for a large number of T values in the range

$5000\,K \leq T \leq 15\,000\,K$

The resulting plots are on the next page

Problem 2(b):



2345X

(3) Phillips problem 2.4.

The dissociation energy is 4.48 eV. The formula analogus to the Saha equation is (correcting Phillips)

$$\frac{n(H)\,n(H)}{n(H_2)} = \left[\frac{\pi m_H kT}{h^2}\right]^{3/2} \exp\left[-\frac{4.48\,\text{eV}}{kT}\right]$$

$$\frac{4.48\,\text{eV}}{k} = 51988\ \text{K}$$ so the exponential is $\exp\left[-\frac{51988}{T}\right]$

Also, if 50% are dissociated, $n(H) = 2n(H_2)$, so

$$\frac{n(H)\,n(H)}{n(H_2)} = 2n(H)$$

also $P = [n(H) + n(H_2)]kT = [n(H) + \frac{1}{2}n(H)]kT = \frac{3}{2}n(H)kT$

Thus $$\frac{n(H)\,n(H)}{n(H_2)} = 2n(H) = \frac{4}{3}\frac{P}{kT}$$

and our equation can be written

$$\exp\left[\frac{51988}{T}\right] = \left[\frac{\pi m_H kT}{h^2}\right]^{3/2} \frac{n(H_2)}{n(H)\,n(H)}$$

$$= \left[\frac{\pi m_H kT}{h^2}\right]^{3/2} \frac{3}{4}\frac{kT}{P}$$

$$\frac{51988}{T} = \log\left[\left[\frac{\pi m_H}{h^2}\right]^{3/2} \frac{3}{4P}(kT)^{5/2}\right]$$

or $$T = 51988\left\{\log\left(\left[\frac{\pi m_H}{h^2}\right]^{3/2}\frac{3}{4P}(kT)^{5/2}\right)\right\}^{-1}$$

In SI units $m_H = 1.67\cdot 10^{-27}$ kg $\quad h = 6.626\cdot 10^{-34}$

$k = 1.38\cdot 10^{-23} \quad P = 100$ Pa

A transendental equation! Iterate. Guess $T_0 = 2000$ K

$$T = \frac{51988}{\log(1.240\cdot 10^{9})} = 2483$$

(3) cont

Put this value of T into the right hand side and solve again:

$$T_2 = \frac{51988}{\log(2.129 \cdot 10^9)} = 2420.41$$

again $$T_3 = \frac{51988}{\log(1.9977 \cdot 10^9)} = 2427.62$$

⋮

and after many iterations it seems to converge to

$$\boxed{T = 2426.86\ K}$$

This corresponds to a number density of $n(H) = \frac{2}{3} P/kT \approx 4.48 \cdot 10^{21}\ m^{-3} = 4.48 \cdot 10^{15}\ cm^{-3}$

M	Pc	Tc	rho_c	R	L/L_sun	conv_1	conv_2	conv_3	T_eff
0.5	7.892(16)	9.604(6)	70.26	3.49(10)	0.0562	0.60798	0.00001	0	3975
0.6	9.311(16)	1.067(7)	73.88	4.03(10)	0.115	0.64714	0.99991	0	4421
0.7	1.093(17)	1.177(7)	77.89	4.61(10)	0.221	0.69209	0.99988	0	4865
0.8	1.276(17)	1.293(7)	82.21	5.30(10)	0.403	0.73724	0.99988	0	5274
0.9	1.477(17)	1.412(7)	86.52	6.26(10)	0.793	0.79323	0.99989	0	5580
1.0	1.683(17)	1.537(7)	90.05	7.41(10)	1.175	0.90165	0.99986	0	5829
1.1	1.853(17)	1.659(7)	91.22	8.21(10)	1.851	0.96618	0.99991	0.04091	6206
1.2	1.944(17)	1.763(7)	89.33	8.46(10)	2.742	0.99211	0.99924	0.06903	6742
1.3	1.961(17)	1.845(7)	85.45	8.62(10)	3.865	0.99305	0.99451	0.08814	7279
1.4	1.932(17)	1.911(7)	80.78	8.81(10)	5.259	0.99227	0.99376	0.10057	7777
1.6	1.817(17)	2.010(7)	71.35	9.26(10)	9.045	0.99282	0.99396	0.11716	8686
1.8	1.682(17)	2.087(7)	63.05	9.76(10)	14.50	0.99229	0.99346	0.12830	9519
2.0	1.554(17)	2.152(7)	56.05	1.03(11)	22.04	0.99325	0.99417	0.13743	10304
2.2	1.437(17)	2.208(7)	50.20	1.08(11)	32.13	0.99323	0.99415	0.14438	11406
2.5	1.286(17)	2.282(7)	43.12	1.16(11)	53.02	0.99424	0.99424	0.15466	12098

Here are the results of running the 15 ZAMS models.

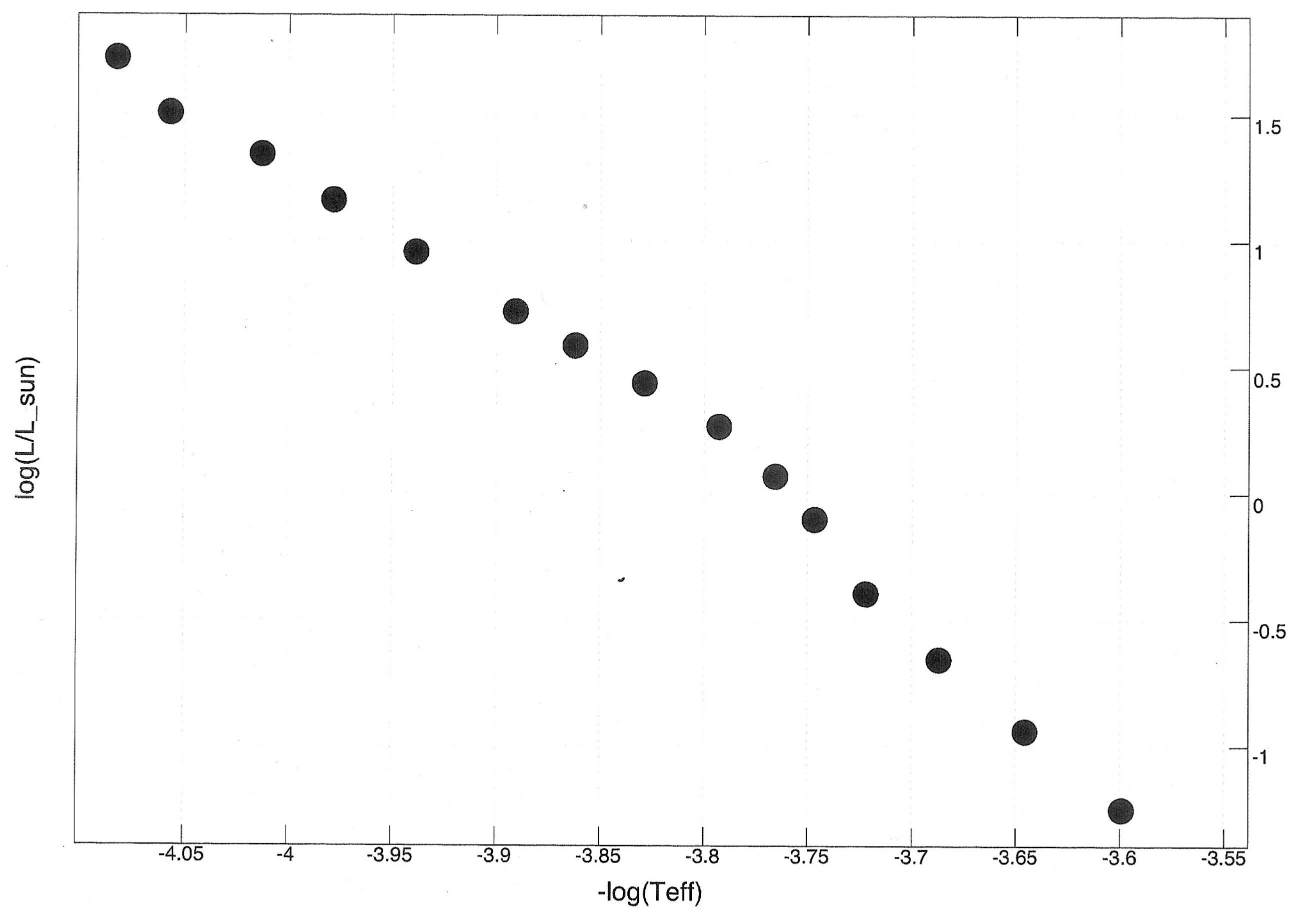
lower upper

boundaries of the outer convection zone

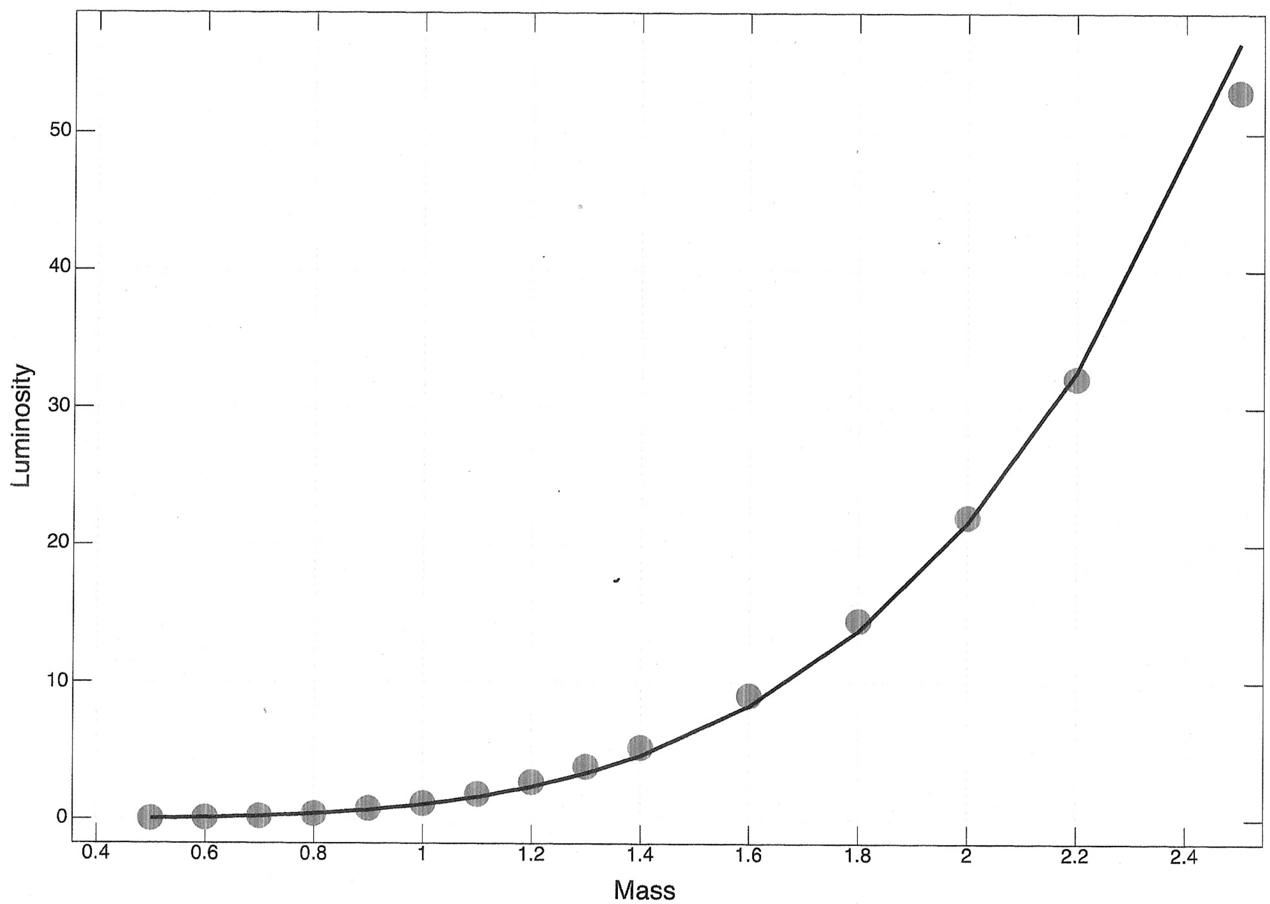
outer boundry of the inner convection zone (inner boundry is at $r=0$).

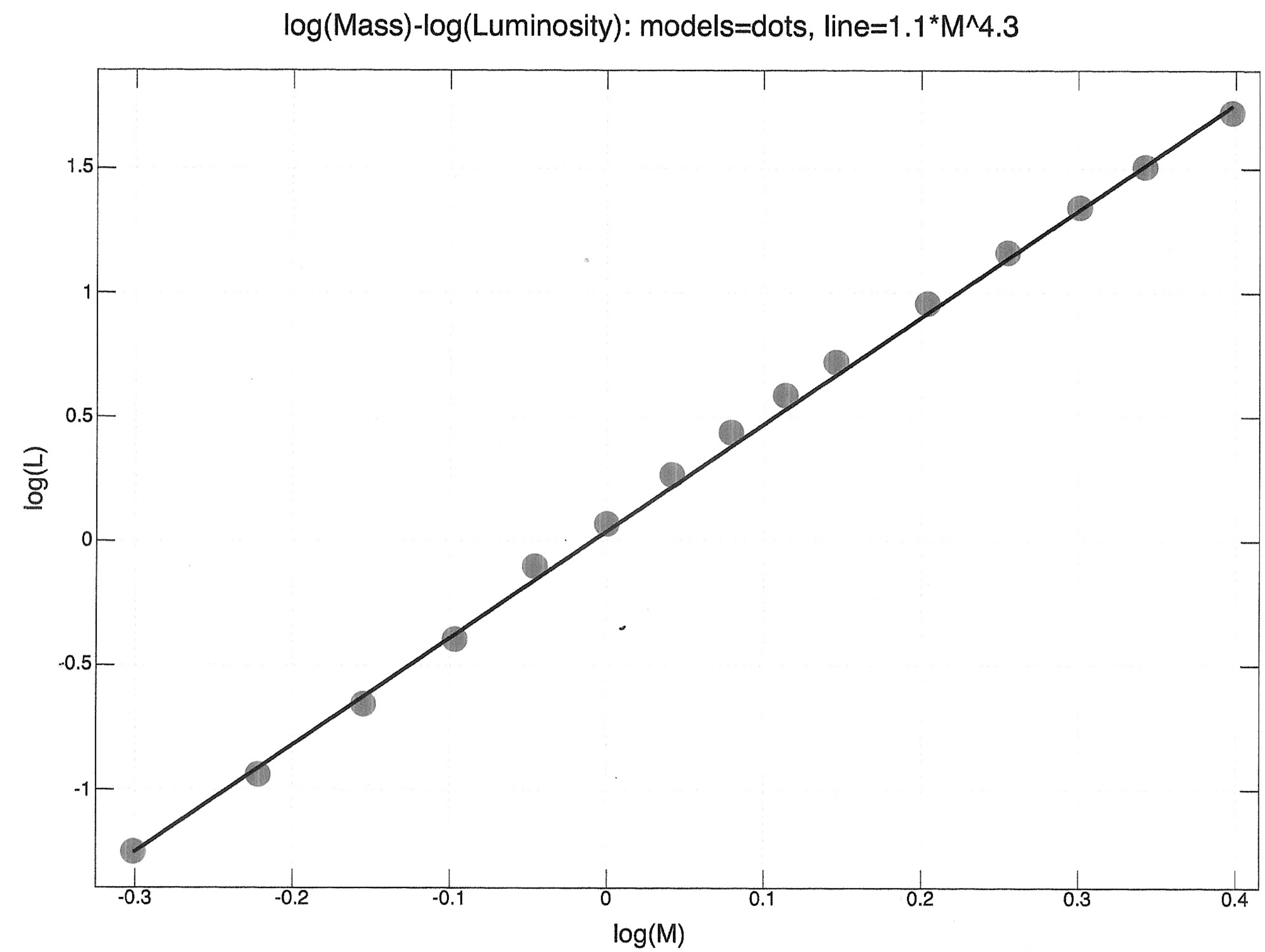
H-R diagram

log(L/L_sun) vs. -log(Teff)

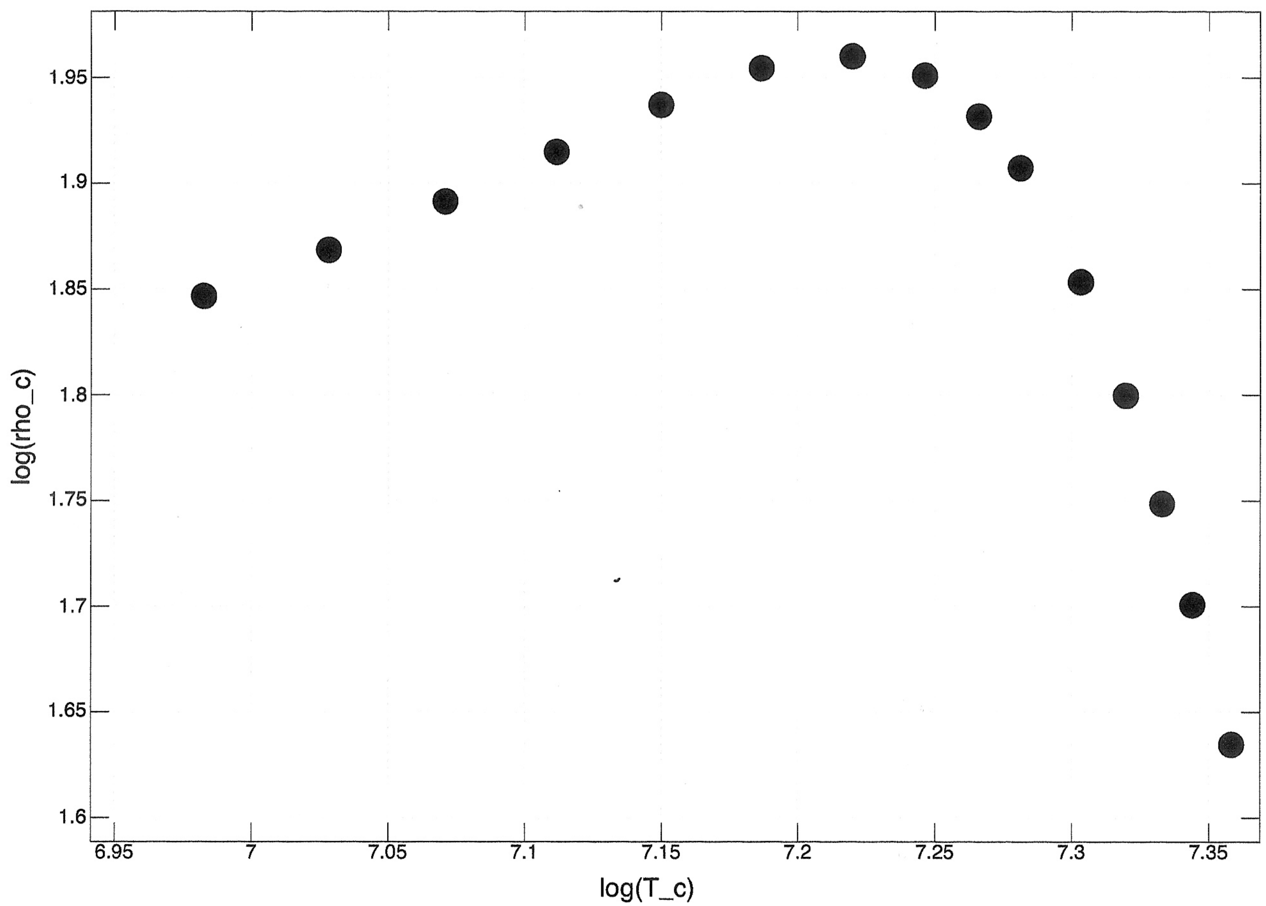


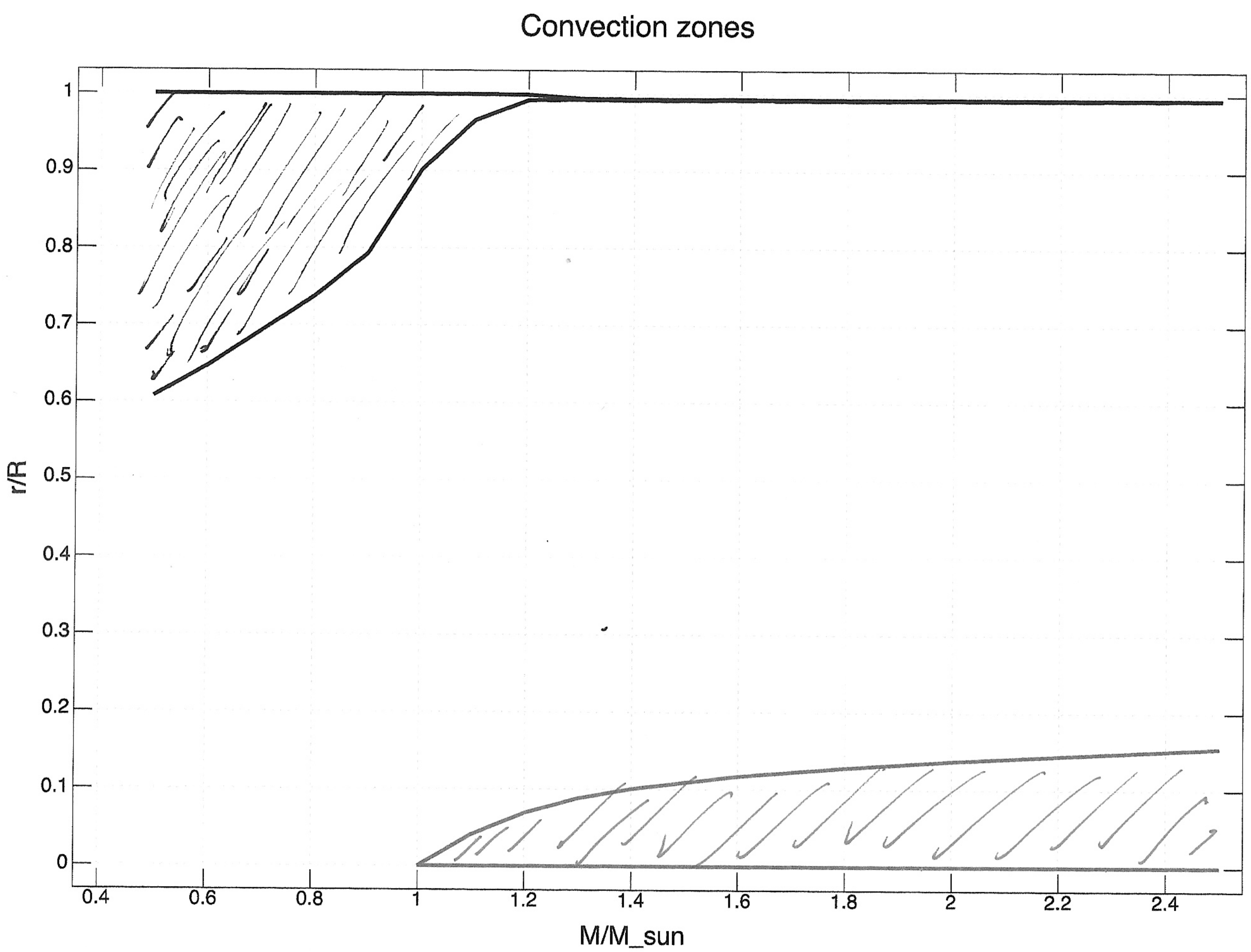
Mass-Luminosity: models=dots, line=1.1*M^4.3




log(Mass)-log(Luminosity): models=dots, line=1.1*M^4.3
log(L)
1.5
1
0.5
0
-0.5
-1
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
log(M)

log(rho_c) vs. log(Tc)




Convection zones
r/R
M/M_sun
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2.2
2.4