1.11. Parabolic Orbits

In the case of zero total energy, $\mathcal{E} = 0$, the orbit is parabolic. Since the eccentricity $e = 1$ while $a = \infty$, the numerator of eqn (76), $a(1 - e^2)$, would seem undefined. But this product can remain finite: we can write the equation of the parabolic conic section as

$$r = \frac{2q}{1 + \cos\theta} \tag{108}$$

It is clear that q represents the point of minimum radius, which occurs at $\theta = 0$. Now in this case eqn (79) is just

$$\mathcal{E} = 0 = \frac{h^2}{2q^2} - \frac{GM}{q} \qquad \text{so that} \qquad h = \sqrt{2GMq} \tag{109}$$

Appealing once again to $\dot{\theta} = h/r^2$, we have

$$\frac{d\theta}{dt} = \frac{\sqrt{2GMq}}{r^2} = \sqrt{2GMq}\,\frac{(1 + \cos\theta)^2}{4q^2} \tag{110}$$

If we let $\phi = \theta/2$, then $(1 + \cos\theta)^2 = (1 + \cos 2\phi)^2 = (\cos^2\phi)^2 = \cos^4\phi$ and this equation becomes

$$\frac{d\phi}{\cos^4\phi} = \sqrt{\frac{GM}{2q^3}}\; dt \tag{111}$$

which can be integrated directly. Let's introduce a variable τ analogous to the mean anomaly:

$$\tau = \sqrt{\frac{GM}{2q^3}}\; (t - t_0) \tag{112}$$

Here, t_0 is the time that the body reaches perihelion ($\theta = 0$, $r = q$). Then the integral of eqn (111) (which is analogous to Kepler's equation) becomes

$$\tau = \frac{1}{3}\frac{\tan\phi}{\cos^2\phi} + \frac{2}{3}\tan\phi = \tan\left(\frac{\theta}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\theta}{2}\right) \tag{113}$$

Unlike Kepler's equation (103), this is a cubic equation and hence we can solve explicitly for θ in terms of time t. The most elegant form introduces two auxiliary variables s and w:

$$\begin{aligned} s &= \arctan[1/\tan(2/3\tau)] \\ w &= \arctan\left\{[\tan(s/2)]^{1/3}\right\} \\ \theta &= 2\arctan\left[2/\tan(2w)\right] \end{aligned} \tag{114}$$

For a body of small mass, like a comet in orbit about the sun, it is convenient to use the Gaussian constant k, so that eqn(112) becomes

$$\tau \ = \ \frac{k}{\sqrt{2q^3}} \ (t - t_0) \tag{115}$$

where q is in AU and t is in days. Then eqns (114) and eqn (108) give us the position of the body at time t.

1.12. Hyperbolic Orbits

When the total energy is positive, $\mathcal{E} > 0$, the orbit is hyperbolic and the eccentricity $e > 1$. When the object is at a great distance, the gravitational potential must go to zero, and all the energy is kinetic. Thus the velocity at large r is $v_\infty = \sqrt{2\mathcal{E}}$. (Recall that $\mathcal{E}$ is the energy per unit mass.) Looking at our previous equation for the ellipse (eqn 76), we see that since $(1 - e^2)$ is negative, if r is to be positive a must be negative. (Some authors take a to be positive and write $a(e^2 - 1)$ for the numerator of eqn 76, but as we will see, it is useful to define $a < 0$.) So we have

$$r \ = \ \frac{(-a)(e^2 - 1)}{1 \ + \ e\cos(\theta)} \tag{116}$$

We see that the closest approach occurs at $\theta = 0$, for which $r_{min} = (-a)(e - 1)$.

In a hyperbolic orbit, the object approaches along one direction θ_{in} , asymptotic to a straight line, is deflected, and recedes asymptotically along another direction θ_{out}. These angles can be found by noting that $r \to \infty$ if $(1 + e\cos\theta) \to 0$. This can only occur for θ in the 2nd and 3rd quadrants, where $\cos\theta$ is negative. If we assume that the orbit is in the direction of increasing θ and that $\theta = 0$ at the point of closest approach, $-\pi/2 < \theta_{in} < -\pi$ and $\pi/2 < \theta_{out} < \pi$. Setting $e\cos\theta = -1$ we obtain

$$\theta_{out} \ = \ \arccos\left(-\frac{1}{e}\right) \qquad \text{and} \qquad \theta_{in} = -\theta_{out} \ \ . \tag{117}$$

For example, if $e = \sqrt{2} = 1.41421$, then $\theta_{in} = -135^o$, $\theta_{out} = 135^o$, and $\Theta = 90^o$. The equations for the asymptotic lines are $y = \pm(e^2 - 1)x$, so in this example $y = \pm x$, and the asymptotic lines make a 45^o angle with the x-axis.

If the gravitational interaction caused no deflection, then we would have $\theta_{out} - \theta_{in} = 2\theta_{out} = \pi$. Thus the total deflection Θ produced by the gravitational interaction must be

$$\Theta = (\theta_{out} - \theta_{in}) - \pi = 2\theta_{out} - \pi = 2\arccos(-1/e) - \pi \tag{118}$$

If we divide eqn (118) by 2 and take the cosine we have

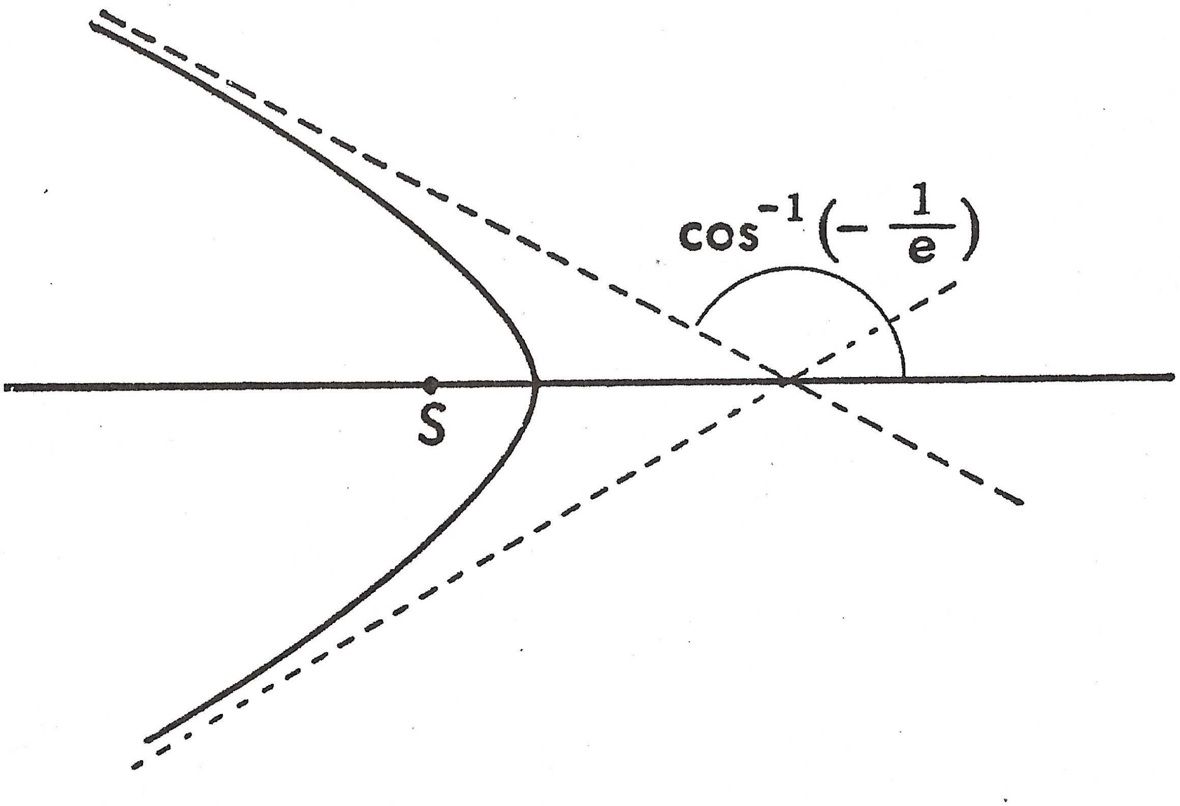

Fig. 1.— A hyperbolic orbit about S and the asymptotic lines.

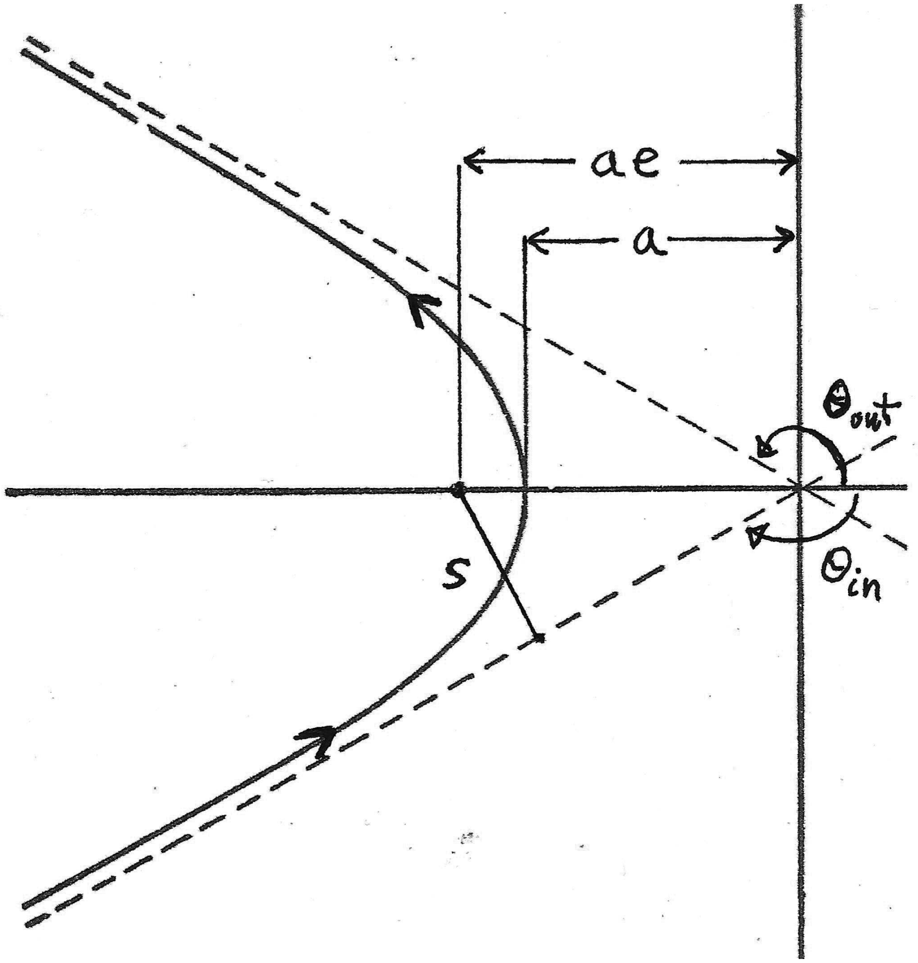

Fig. 2.— A hyperbolic orbit showing s, the impact parameter. It is perpendicular to the asymptotic line.

$$\cos\left(\frac{\Theta}{2} + \frac{\pi}{2}\right) = -\frac{1}{e} \qquad \text{or} \qquad \sin\left(\frac{\Theta}{2}\right) = \frac{1}{e} \tag{119}$$

Now, lets return to the *vis viva* equation (88), remembering that a is negative:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) = GM\left(\frac{2}{r} + \frac{1}{(-a)}\right) \quad , \tag{120}$$

and the second term is positive (which is why we chose to adopt negative a). Now note that as $r \to \infty$ the first term vanishes and we can write $v_\infty^2 = GM/(-a)$, where v_∞ represents the velocity of the object far from the origin. This sets the value of a:

$$(-a) = \frac{GM}{v_\infty^2} \qquad \text{and we can write} \qquad v^2 = v_\infty^2 + \frac{2GM}{r} \quad , \tag{121}$$

which shows how $v(r)$ increases as the body approaches the central attractor. The maximum velocity v_m will occur at the minimum radius r_m. Using $r_m = (-a)(e-1)$ and $(-a)$ from above, we see that

$$r_m = \frac{GM}{v_\infty^2}(e-1) \qquad \text{and} \qquad v_m^2 = v_\infty^2\left(\frac{e+1}{e-1}\right) \quad . \tag{122}$$

At some very great distance the particle will be traveling along a nearly straight path, the asymptotic line. If the particle were not deflected, it would pass by the attracting center at some distance called the *impact parameter*, s . From the second diagram, we can see the triangle with side s, hypotenuse (-ae), and angle $(\pi - \theta_{out})$. From this triangle we get the relation $s = (-ae)\sin(\pi - \theta_{out}) = (-ae)\sin(\theta_{out})$. Since $\cos(\theta_{out}) = -1/e$, $\sin(\theta_{out}) = (1 - \cos^2(\theta_{out}))^{1/2} = (1 - 1/e^2)^{1/2}$ and thus we have the relation

$$s = (-ae)\left(1 - \frac{1}{e^2}\right)^{1/2} = (-a)(e^2 - 1)^{1/2} = \frac{GM}{v_\infty^2}(e^2 - 1)^{1/2} \tag{123}$$

Solving this for the eccentricity, we have

$$e^2 = 1 + \left(\frac{v_\infty^2\, s}{GM}\right)^2 \tag{124}$$

Hyperbolic orbits are useful in treating the accretion of material by planets. If we set the radius of the planet equal to the point of closest approach, $r_m = R_p$, then we see that any particle approaching with an impact parameter less that the corresponding s from eqn (123) will strike the planet. But to evaluate this *effective cross section*, πs^2 , we would like to express s in terms of v_∞ and r_m . Now from eqn (122) we have another expression for e^2:

$$e^2 \;=\; \left[1 \;+\; \frac{v_\infty^2 \; r_m}{GM}\right]^2 \tag{125}$$

Equate (124) and (125) and define $\alpha = v_\infty^2/GM$ and we find

$$\begin{aligned} (1+\alpha r_m)^2 \;&=\; 1 + (\alpha s)^2 \\ \alpha^2 s^2 \;&=\; \alpha^2 r_m^2 \;+\; 2\alpha r_m \\ s^2 \;&=\; r_m^2(1 + 2/\alpha r_m) \end{aligned} \tag{126}$$

and setting $r_m = R_p$, the radius of the accreting planet, we get the effective cross section for accretion πs^2 in the form

$$\pi s^2 \;=\; \left[1 + \frac{2GM}{R_p v_\infty^2}\right] \; \pi R_p^2 \tag{127}$$

We see that the geometrical cross section πR_p^2 is enhanced by a factor of one plus the quantity

$$\frac{2GM}{R_p v_\infty^2} \;=\; \frac{\frac{GM}{R_p}}{\frac{1}{2}v_\infty^2} \;=\; \left(\frac{v_{esc}}{v_\infty}\right)^2 \tag{128}$$

which is just the ratio of the potential energy at the surface of the planet to the kinetic energy of the particle at infinity. It is also the square of the ratio of the escape velocity from the planet to the velocity at infinity.

Another result that will prove useful follows from eqn (119) for the deflection angle when combined with eqn (124) for e^2. We use the trigonometric identity $1 + \cot^2(x) = 1/\sin^2(x)$. This leads to

$$\cot^2\left(\frac{\Theta}{2}\right) \;=\; e^2 \;-\; 1 \tag{129}$$

so that

$$\cot\left(\frac{\Theta}{2}\right) \;=\; \left(\frac{v_\infty^2 \; s}{GM}\right) \tag{130}$$

which gives us the deflection angle in terms of the velocity of the incoming particle and its impact parameter.

For completeness we should include the equations to find the position of the object as a function of time. The equations are analogous to equations (99), (100) and (103), except that hyperbolic trigonometric functions are involved:

$$r \;=\; (-a) \;\; (e \;\; \cosh E \;-\; 1) \tag{131}$$

$$\tan\left(\frac{\theta}{2}\right) \;=\; \sqrt{\frac{e+1}{e-1}} \;\; \tanh\left(\frac{E}{2}\right) \tag{132}$$

$$M \;=\; e \;\; \sinh E \;-\; E \tag{133}$$