1. Review of Mechanics

1.1. Newton's Laws

Motion of particles. Let the position of the particle be given by $\vec{r}$. We can always express this in Cartesian coordinates:

$$\vec{r} \;=\; x\hat{x} \;+\; y\hat{y} \;+\; z\hat{z} \quad , \tag{1}$$

where we will always use ˆ (circumflex) to represent a unit vector. Or we could use spherical or cylindrical coordinates. For a while, we will deal mostly with motion in a fixed plane. We then can locate the particle by the magnitude r ($\vec{r} \;=\; r\hat{r}$) and an angle θ.

As the particle moves, the time derivative of the (vector) position is the velocity, $\vec{v} = d\vec{r}/dt = \dot{\vec{r}}$.

$$\frac{d\vec{r}}{dt} \;=\; \frac{d}{dt}(r\hat{r}) \;=\; \frac{dr}{dt}\,\hat{r} \;+\; r\,\frac{d\hat{r}}{dt} \quad . \tag{2}$$

Now it is easy to see that the derivatives of the unit vectors are given by

$$\frac{d\hat{r}}{dt} \;=\; \frac{d\theta}{dt}\,\hat{\theta} \qquad \textit{and} \qquad \frac{d\hat{\theta}}{dt} \;=\; -\,\frac{d\theta}{dt}\,\hat{r} \tag{3}$$

so the velocity in polar coordinates is given by

$$\vec{v} \;=\; \dot{\vec{r}} \;=\; \frac{dr}{dt}\,\hat{r} \;+\; r\,\frac{d\theta}{dt}\,\hat{\theta} \quad . \tag{4}$$

Consider a particle in a circular orbit about the origin. Then $dr/dt = 0$, but $d\theta/dt = \omega$, the angular velocity, and $\vec{v} = r\dot{\theta}\hat{\theta} \neq 0$, and directed perpendicular to the radius vector. From the velocity, we form the *momentum*, $\vec{p} = m\vec{v}$. Usually, the mass is constant, and we will assume that unless we say otherwise.

We also need to consider the change in $\vec{v}$ with time, the *acceleration*, $\vec{a}$:

$$\begin{aligned}
\vec{a} \;&=\; \dot{\vec{v}} \;=\; \ddot{\vec{r}} \;=\; \frac{d}{dt}\left(\frac{dr}{dt}\,\hat{r} \;+\; r\,\frac{d\theta}{dt}\,\hat{\theta}\right) \\
&=\; \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt} \\
&=\; \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} - r\left(\frac{d\theta}{dt}\right)^2\hat{r} \\
&=\; \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\theta} \\
\ddot{\vec{r}} \;&=\; \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \;\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}
\end{aligned} \tag{5}$$

Newton's second law is

$$\vec{F} \;=\; \frac{d\vec{p}}{dt} \;=\; m\,\frac{d\vec{v}}{dt} \;=\; m\,\vec{a} \quad . \tag{6}$$

This is a **vector** equation, i.e., it implies three scalar equations:

$$F_x = ma_x, \qquad F_y = ma_y, \qquad F_z = ma_z \quad .$$

Now suppose we have two particles. Let the force on (1) exerted by (2) be called $\vec{F}_{12}$. Newton's third law states that the force exerted on (2) by (1) is given by $\vec{F}_{21} = -\vec{F}_{12}$. Then from Newton's second law we have

$$m_1\vec{a}_1 \;=\; -\,m_2\vec{a}_2 \qquad ,\text{i.e.,} \qquad \vec{a}_2 \;=\; -\,\frac{m_1}{m_2}\vec{a}_1 \quad . \tag{7}$$

Thus if $m_2 >> m_1$, then $|\vec{a}_2| << |\vec{a}_1|$; for example, though an orbiting satellite exerts as much force on the earth as the earth exerts on the satellite, the acceleration of the earth is much less than that of the satellite.

1.2. Angular Momentum

The angular momentum of a particle is defined as $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$, where $\times$ denotes the vector cross product. The order matters, as $\vec{p} \times \vec{r} = -\vec{L}$. The vector $\vec{L}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{v}$. In general, the cross product is given by

$$\begin{aligned} \vec{a} \times \vec{b} \;&=\; |a| \cdot |b|\ \sin(\gamma)\ \hat{n} \\ &=\; \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_z & b_y & b_z \end{vmatrix} \\ &=\; (a_y b_z - a_z b_y)\hat{x} - (a_x b_z - a_z b_x)\hat{y} + (a_x b_y - a_y b_x)\hat{z} \end{aligned}$$

Here, γ is the angle between the two vectors and $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. The expression on the second line should be expanded as the determinant of the array.

Also, the *torque* $\vec{N}$ associated with the force $\vec{F}$ is defined as

$$\vec{N} \;=\; \vec{r} \times \vec{F} \tag{8}$$

Note that both $\vec{N}$ and $\vec{L}$ depend upon the choice of the coordinate origin.

Now if we cross $\vec{r}$ into both sides of Newton's second law we get

$$\vec{N} \;=\; \vec{r} \times \vec{F} \;=\; \vec{r} \times \frac{d\vec{p}}{dt} \quad . \tag{9}$$

Next, note that

$$\frac{d\vec{L}}{dt} \;=\; \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) \;=\; \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} \;=\; \vec{v} \times m\vec{v} + \vec{r} \times \frac{d\vec{p}}{dt} \;=\; \vec{r} \times \frac{d\vec{p}}{dt} \tag{10}$$

and we find that the time derivative of the angular momentum is equal to the applied torque:

$$\vec{N} \;=\; \frac{d\vec{L}}{dt} \tag{11}$$

In the absence of torques, $\vec{N} = 0$, the angular momentum $\vec{L}$ is constant.

1.3. Systems of Particles

Suppose we have a (somewhat) isolated system of particles. It is useful to distinguish between internal and external forces. Then Newton's 2nd law for the i^{th} particle is

$$\frac{d\vec{p}_i}{dt} \;=\; \sum_{j \neq i} \vec{F}_{ij} \;+\; \vec{F}_i^{(e)} \tag{12}$$

Here, $\vec{F}_i^{(e)}$ is the external force on the i^{th} particle. Summing this expression over all the particles, we have

$$\frac{d}{dt} \sum_i m_i \frac{d\vec{r}_i}{dt} \;=\; \sum_i \sum_{j \neq i} \vec{F}_{ij} \;+\; \sum_i \vec{F}_i^{(e)} \tag{13}$$

Now since for every $\vec{F}_{ji}$ there is a corresponding $\vec{F}_{ij} = -\vec{F}_{ji}$, the double sum must vanish, leaving us with

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i \;=\; \sum_i \vec{F}_i^{(e)} \tag{14}$$

The total mass of our system of particles is $M = \sum_i m_i$. We define the *center of mass* (CM) as

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{M} \tag{15}$$

Then eqn(14) becomes

$$M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)} \tag{16}$$

where $\vec{F}^{(e)}$ is the total external force on the system. We see that the CM of the system moves as if it were a particle of mass M acted on by $\vec{F}^{(e)}$. We also define the *total linear momentum* as

$$\vec{P} = M\frac{d\vec{R}}{dt} \tag{17}$$

so that we may write an equation for the whole system that looks like Newton's 2^{nd} law:

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)} \tag{18}$$

If there is no external force, then $\vec{P} =$ constant: the total linear momentum of the system is conserved.

Next, let's look at the angular momentum of the system. We cross $\vec{r}_i$ into eqn(12) and sum over the particles to obtain

$$\sum_i \left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right) = \sum_i \sum_{j\neq i} \vec{r}_i \times \vec{F}_{ij} + \sum_i \vec{r}_i \times \vec{F}_i^{(e)} \tag{19}$$

Now the double sum is the sum of pairs of torques:

$$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji} = (\vec{r}_j - \vec{r}_i) \times \vec{F}_{ji} \tag{20}$$

But the vector $(\vec{r}_j - \vec{r}_i)$ is along the line between the two particles, as is the force $\vec{F}_{ij}$. Thus the cross product is zero for all ij pairs, and the double sum must vanish. As we saw in eqn(10), we can write the left hand side of eqn(19) as

$$\sum_i \left(\vec{r}_i \times \frac{d\vec{p}_i}{dt}\right) = \frac{d}{dt}\sum_i (\vec{r}_i \times \vec{p}_i) \tag{21}$$

Let us define the *total angular momentum* of the system of particles as $\vec{L} = \sum_i (\vec{r}_i \times \vec{p}_i)$ and the *total external torque* as $\vec{N}^{(e)} = \sum_i \left(\vec{r}_i \times \vec{F}_i^{(e)}\right)$. Then we see that eqn(19) reduces to

$$\frac{d\vec{L}}{dt} = \vec{N}^{(e)} \tag{22}$$

In the case where the external torque, $\vec{N}^{(e)}$, vanishes, we see that $\vec{L}$ = constant: the total angular momentum of the system is conserved.

Now the value of $\vec{L}$ depends upon the choice of the coordinate system. Let $\vec{r_i'}$ refer to the points relative to the CM (center of mass), whose position is $\vec{R}$:

$$\vec{r_i} = \vec{R} + \vec{r_i'} \tag{23}$$

Also, let the velocity of the CM be $\vec{V} = d\vec{R}/dt$, so that

$$\vec{v_i} = \vec{V} + \vec{v_i'} \qquad \text{where} \qquad \vec{v_i'} = \frac{d\vec{r_i'}}{dt} \tag{24}$$

Then it is easy to show (!) that

$$\vec{L} = \vec{R} \times \vec{P} + \sum_i \left(\vec{r_i'} \times \vec{p_i'} \right) \tag{25}$$

I.e., the total angular momentum is the sum of the angular momentum of the particles referenced to the center of mass, plus a second term which is the angular momentum of a mass M moving with the CM.

Now, **if** the CM of the system is at rest in the original coordinate system, then $\vec{P} = M\vec{V} = 0$, and $\vec{L}$ is just the angular momentum relative to the center of mass, *regardless to the origin.*

A very important quantity is the *total kinetic energy* of our system of particles:

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \; \vec{v} \cdot \vec{v} \qquad (v_i = |\vec{v_i}|) \tag{26}$$

Now $\vec{v_i} = \vec{V} + \vec{v_i'}$, so $\vec{v} \cdot \vec{v} = V^2 + 2\vec{V} \cdot \vec{v_i'} + (v_i')^2$. Thus

$$T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i (v_i')^2 + \sum_i m_i \vec{V} \cdot \frac{d\vec{r_i'}}{dt} \tag{27}$$

But the argument of the last sum is $\vec{V} \cdot \frac{d}{dt}\sum_i m_i \; \vec{r_i'}$, and this sum vanishes by definition of the CM. ($\sum_i m_i \; \vec{r_i'} = \sum_i m_i \; \vec{r_i} - \sum_i m_i \; \vec{R} = M\vec{R} - M\vec{R} = 0$.)

So the total kinetic energy of the system has two parts, just like the angular momentum:

$$T \;=\; \frac{1}{2}MV^2 \;+\; \frac{1}{2}\sum_i m_i(v_i')^2 \tag{28}$$

1.4. Reduction of the Two-Body System

Now we want to look at a system of only two masses, with no external forces.

The total mass is $M = m_1 + m_2$ and the center of mass is located at

$$\vec{R} \;=\; \frac{1}{M}\sum_i m_i\vec{r_i} \;=\; \frac{m_1}{M}\vec{r_1} \;+\; \frac{m_2}{M}\vec{r_2} \tag{29}$$

Now $\vec{r_1'} = \vec{r_1} - \vec{R}$, so

$$\vec{r_1'} = \vec{r_1} - \frac{m_1}{M}\vec{r_1} - \frac{m_2}{M}\vec{r_2} \;=\; \frac{(m_1+m_2)-m_1}{M}\vec{r_1} - \frac{m_2}{M}\vec{r_2} \;=\; \frac{m_2}{M}\left(\vec{r_1}-\vec{r_2}\right) \tag{30}$$

Define $\vec{r} \;=\; \vec{r_2}-\vec{r_1}$, which is the the vector pointing *from* $\vec{r_1}$ towards $\vec{r_2}$. Also, treating $\vec{r_2'}$ in the same way, we get the following relations:

$$\vec{r_1'} \;=\; -\,\frac{m_2}{M}\,\vec{r} \qquad\qquad \vec{r_2'} \;=\; \frac{m_1}{M}\,\vec{r} \tag{31}$$

Next, let us define the **reduced mass**, μ :

$$\mu \;=\; \frac{m_1m_2}{M} \;=\; \frac{m_1m_2}{m_1+m_2} \qquad\text{or}\qquad \frac{1}{\mu} \;=\; \frac{1}{m_1} \;+\; \frac{1}{m_2} \tag{32}$$

then

$$\vec{r_1'} \;=\; -\,\frac{\mu}{m_1}\,\vec{r} \qquad\qquad \vec{r_2'} \;=\; \frac{\mu}{m_2}\,\vec{r} \tag{33}$$

We see that the one vector, $\vec{r}$, is sufficient to specify the position of both particles.

Next, let us consider the kinetic energy of the system. We will assume that the CM is at rest in the coordinate system, so $\vec{V} = 0$, and if there are no external forces, then by eqn(18), it will remain so. Then from eqn(28),

$$T \quad=\quad \frac{1}{2}m_1(v_1')^2 \;+\; \frac{1}{2}m_2(v_2')^2$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{m_1\left[\frac{d}{dt}\left(-\frac{\mu}{m_1}\vec{r}\right)\right]^2 + m_2\left[\frac{d}{dt}\left(\frac{\mu}{m_2}\vec{r}\right)\right]^2\right\} \\
&= \frac{\mu^2}{2}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]\left(\frac{d\vec{r}}{dt}\right)^2 \\
T &= \frac{1}{2}\,\mu\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{1}{2}\,\mu\,v^2 \qquad (34)
\end{aligned}
$$

We see that the kinetic energy of the system looks like that of a single particle of mass μ .

Since we will be dealing with motion in a plane, we would like the kinetic energy in polar coordinates. Making use of eqn(4), we see that

$$
v^2 = \vec{v}\cdot\vec{v} = \left[\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right]\cdot\left[\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right] = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 \qquad (35)
$$

and using the "dot" notation, the total kinetic energy of the 2-body system is just

$$
T = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \qquad (36)
$$

Note that the reduced mass is closer to (but smaller than) the *less massive* of the two particles:

$$
\begin{aligned}
m_2 = 100\ m_1 &\quad ------> \quad \mu = 0.99\ m_1 \\
m_2 = 10\ m_1 &\quad ------> \quad \mu = 0.91\ m_1 \\
m_2 = m_1 &\quad ------> \quad \mu = 0.5\ m_1
\end{aligned}
$$

Next, we can evaluate the angular momentum of the system. From eqn(25) we have

$$
\vec{L} = \left(\vec{r_1'} \times m_1\frac{d\vec{r_1'}}{dt}\right) + \left(\vec{r_2'} \times m_2\frac{d\vec{r_2'}}{dt}\right) \qquad (37)
$$

Using eqn(33) to eliminate $\vec{r_1'}$ and $\vec{r_2'}$ we have

$$
\begin{aligned}
\vec{L} &= \left[\left(-\frac{\mu}{m_1}\vec{r}\right)\times m_1\left(-\frac{\mu}{m_1}\right)\frac{d\vec{r}}{dt}\right] + \left[\left(\frac{\mu}{m_2}\vec{r}\right)\times m_2\left(\frac{\mu}{m_2}\right)\frac{d\vec{r}}{dt}\right] \\
&= \mu^2\left[\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{r}\times\frac{d\vec{r}}{dt}\right] \\
\vec{L} &= \mu\vec{r}\times\frac{d\vec{r}}{dt} = \vec{r}\times\mu\vec{v} \qquad (38)
\end{aligned}
$$

Once again, the result looks just like a single particle of mass μ at position $\vec{r}$. Now since $\vec{N^{(e)}} = 0$, $\vec{L}$ is constant and perpendicular to the plane including $\vec{r}$ and $\vec{v}$. This means that the motion must remain in the same plane. We can then choose the direction of $\vec{L}$ as the z-axis of a cylindrical coordinate system, and describe the motion of $\vec{r}$ by the coordinates r and θ. Then, using eqn(4), the total angular momentum is

$$\vec{L} = (r\,\hat{r}) \times \mu(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}) \tag{39}$$

Since $\hat{r} \times \hat{r} = 0$ and $\hat{r} \times \hat{\theta} = \hat{z}$, we have

$$\vec{L} = \mu\, r^2\, \dot{\theta}\, \hat{z} = \text{constant} \tag{40}$$

If we define the angular momentum per unit mass as $h = |\vec{L}|/\mu$, then we have that

$$\boxed{h = r^2\, \dot{\theta} = \text{constant}} \tag{41}$$

Eqn(41) is an integral of the motion of the 2-body system and just expresses the conservation of angular momentum. Note that we have as yet not specified the force between the two particles.

If we consider the $\vec{r}$ vector as it moves, the tip of the vector is displaced by a distance $r\ d\theta$ in the $\hat{\theta}$ direction. We see that the area between the old and new positions of $\vec{r}$ amounts to $dS = (1/2)r(rd\theta)$. (The change in area caused by the displacement in the $\hat{r}$ direction will be proportional to $drd\theta$ and thus vanishes in the limit.)

We thus see that the rate at which the radius vector sweeps out area is dS/dt, which by eqn(41) is constant:

$$\frac{dS}{dt} = \frac{1}{2}\, r^2\, \frac{d\theta}{dt} = \frac{1}{2}\, h = \text{constant} \tag{42}$$

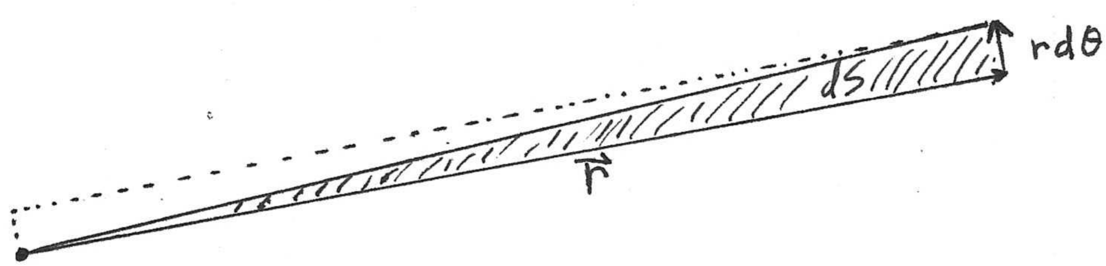


Fig. 1.— The radius vector sweeps out area dS.

and the area swept out in any finite interval of time Δt must be

$$\Delta S = \frac{1}{2} h \, \Delta t \tag{43}$$

This is just **Kepler's 2^{nd} law**. Note that this result does not depend upon the inverse square nature of the gravitational force. It is just an expression of the conservation of angular momentum and applies to any central force motion.

1.5. Newton's Law of Gravitation

Newton's law of universal gravitation states that between any two masses, there is a force exerted on the first, m_1 , due to the second, m_2 , given by

$$\vec{F}_{12} \; = \; \frac{G \; m_1 m_2}{r^2} \, \hat{r} \qquad \text{where} \qquad \vec{r} = \vec{r_2} - \vec{r_1} \; , \qquad \hat{r} = \vec{r}/r \tag{44}$$

and by Newton's third law, the force on the second mass, m_2 , due to m_1 , is just $-\vec{F}_{12}$. Inserting this force into Newton's 2^{nd} law yields

$$m_1 \frac{d^2 \vec{r_1}}{dt^2} \; = \; \frac{d\vec{p_1}}{dt} \; = \; \vec{F}_{12} \; = \; \frac{G m_1 m_2}{r^2} \, \hat{r} \tag{45}$$

where we have placed the origin at the CM so that $\vec{r_1} = \vec{r_1'}$. Using eqn(33) to replace $\vec{r_1}$ by $\vec{r}$, we have

$$\left[\frac{m_1 \; m_2}{M} \right] \frac{d^2 \vec{r}}{dt^2} \; = \; - \, \frac{G \; m_1 m_2}{r^2} \, \hat{r} \tag{46}$$

Thus

$$\ddot{\vec{r}} \; = \; - \, \frac{G \; M}{r^2} \, \hat{r} \qquad \text{or we can write} \qquad \mu \, \ddot{\vec{r}} \; = \; - \, \frac{G \; M \mu}{r^2} \, \hat{r} \tag{47}$$

where the second expression shows explicitly that the equation of motion of either of the two bodies is equivalent to the equation of motion of a mass μ orbiting a *fixed* mass M at distance $\vec{r}$. Using eqn(5) to write $\ddot{\vec{r}}$ in polar coordinates, we have the vector equation

$$(\ddot{r} - r\dot{\theta}^2)\hat{r} \; + \; (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \; = \; - \, \frac{G \; M}{r^2} \, \hat{r} \tag{48}$$

The $\hat{\theta}$ component of this equation is just $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$. But

$$2\dot{r}\dot{\theta} + r\ddot{\theta} \; = \; \frac{d}{dt}\left(r^2 \dot{\theta} \right) \; = \; 0 \tag{49}$$

which integrates immediately to $r^2\dot{\theta} = h$, i.e., just eqn(41), which we already obtained from angular momentum conservation. Turning to the $\hat{r}$ component of eqn(48), we find

$$\ddot{r} \; = \; r\dot{\theta}^2 \; - \frac{G\;M}{r^2} \; = \; \frac{h^2}{r^3} \; - \frac{G\;M}{r^2} \tag{50}$$

where we have used eqn(41) to eliminate $\dot{\theta}$. This is the basic differential equation in r that we must solve to determine the orbit.

1.6. The Gravitational Potential and the Lagrangian of the Problem

Before we integrate eqn(50) for the shape of the orbit, it is useful to take a more general approach which will give us some insight all central force problems. We first note that the *gravitational potential energy* of a mass M is given by

$$V(r) \; = \; - \frac{G\;M}{r} \tag{51}$$

Now the force is given by the gradient of the potential,

$$f(r) \; = \; - \nabla V \tag{52}$$

The gradient acting on a scalar function generates a vector. In the cylindrical coordinates we are using for this problem, the gradient is

$$\nabla \; = \; \hat{r}\,\frac{\partial}{\partial r} \; + \; \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} \; + \; \hat{z}\,\frac{\partial}{\partial z} \tag{53}$$

so in this case, where the potential depends only upon r we have

$$f(r) \; = \; - \nabla V(r) \; = \; - \nabla \left\{ - \frac{G\;M}{r} \right\} \; = \; - \frac{G\;M}{r^2}\,\hat{r} \tag{54}$$

which is just the Newtonian gravitational force (per unit mass, since we did not include μ in the equation).

The *Lagrangian*, $\mathcal{L}$, for a system is the kinetic minus the potential energy. For our central force problem it is thus

$$\mathcal{L} \; = \; T \; - V(r) \; = \; \frac{1}{2}(\dot{r}^2 + r^2\,\dot{\theta}^2) \; + \; \frac{G\;M}{r} \tag{55}$$

A more profound approach to mechanical problems is by means of *Hamilton's Principle*, which states that the motion of a system from time t_1 to time t_2 is such that the *action*, defined as

$$\mathcal{A} \;=\; \int_{t_1}^{t_2} \mathcal{L} \;\; dt \tag{56}$$

is a minimum. It can then be shown that this principle leads directly to the *Euler-Lagrange equations of motion*:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} \;=\; 0 \qquad\qquad i = 1, 2, ..., n \tag{57}$$

where the q_i are the (generalized) coordinates and the $\dot{q}_i$ the corresponding velocities. In our problem, $q_1 \;=\; r$ and $q_2 \;=\; \theta$. Operating on eqn(55) with eqn(57), we immediately obtain

$$\frac{d}{dt}\left(\dot{r}\right) - \left(r\dot{\theta}^2 - \frac{G\;M}{r^2}\right) \;=\; 0 \qquad \text{and} \qquad \frac{d}{dt}\left(r^2\dot{\theta}\right) \;=\; 0 \tag{58}$$

which are precisely eqn(49) and eqn(50) that we obtained from the Newtonian equations.

While the Lagrangian is the difference between the kinetic and potential energies, it is useful to look at their sum, the total energy. $\mathcal{E}$, which must be constant:

$$\mathcal{E} \;=\; T \;+\; V(r) \;=\; \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) + V(r) \tag{59}$$

where we have used $\dot{\theta} = h/r^2$ from eqn(41).

1.7. The Equivalent One-Dimensional Problem

The expression for the total energy, which is conserved, can give us a useful way of looking at the central force problem. Suppose the central force $f(r)$ can be derived from a potential $V(r)$ such that $f(r) = -\partial V/\partial r$. Now let us introduce a *fictitious potential*, $V'(r)$:

$$V'(r) \;=\; V(r) \;+\; \frac{h^2}{2r^2} \qquad \text{so that} \qquad f'(r) \;=\; -\frac{\partial V'}{\partial r} \;=\; f(r) + \frac{h^2}{r^3} \tag{60}$$

in the case of Newtonian gravity, $f(r) = -GM/r^2$, so that

$$V'(r) \;=\; -\frac{GM}{r} \;+\; \frac{h^2}{2r^2} \qquad \text{and thus} \qquad f'(r) \;=\; \frac{h^2}{r^3} \;-\; \frac{G\;M}{r^2} \tag{61}$$

But this means that eqn(50) is just

$$\ddot{r} \;=\; f'(r) \tag{62}$$

This is the equation for a particle moving in one dimension under a force $f'(r)$. The total energy of this particle, in terms of its *radial* velocity $v = \dot{r}$, is

$$\mathcal{E} \;=\; \frac{1}{2}v^2 \;+\; V'(r) \qquad \text{or} \qquad \frac{1}{2}v^2 \;=\; \mathcal{E} \;-\; V'(r) \tag{63}$$

If we than plot the curve $V'(r)$ vs r, and draw the horizontal line which represents the constant value $\mathcal{E}$, we see that the distance of $\mathcal{E}$ above $V'(r)$ gives the kinetic energy and hence v at that r. Where the curves intersect, v must go to zero; this is a turning point of the motion. The particle cannot move to any value of r where the $V'(r)$ curve is above $\mathcal{E}$, for that would require negative kinetic energy, and hence imaginary velocity. Analysis of the fictitious potential eqn(61) shows that we can have either bound orbits or orbits that escape to infinity. However, as long as $h > 0$, a particle cannot reach the origin.

Now consider the force as given by Einstein's general theory of relativity. Then for a mass M there is a critical radius, the *Schwarzschild radius*, given by $r_S = 2GM/c^2$. In terms of this radius, the fictitious potential is

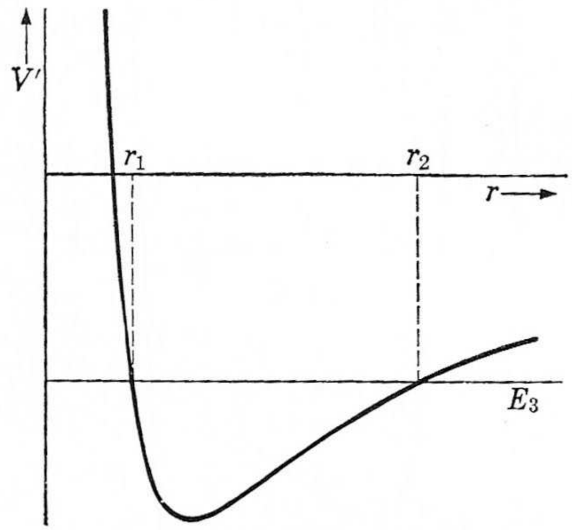
$$V'(r) \;=\; (1 - r_S/r)\left(\frac{c^2}{2} + \frac{h^2}{2r^2}\right) \;=\; \frac{c^2}{2} \;-\; \frac{GM}{r} \;+\; \frac{h^2}{2r^2} \;-\; \frac{GMh^2}{c^2r^3} \tag{64}$$

which corresponds to a force

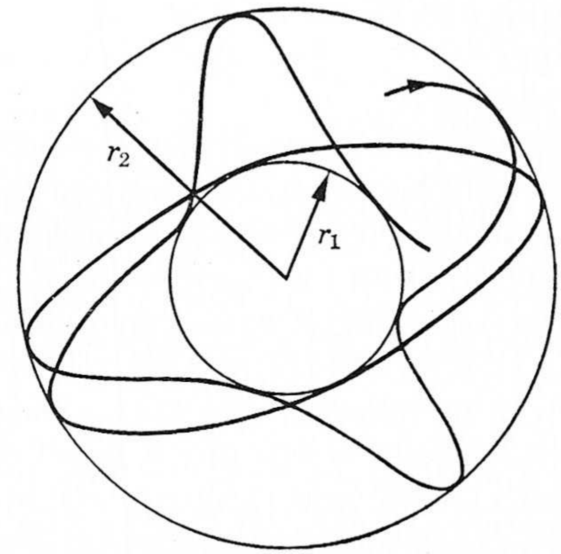
$$f'(r) \;=\; -\frac{GM}{r^2} \;+\; \frac{h^2}{r^3} \;-\; \frac{3GMh^2}{c^2r^4} \tag{65}$$

The $c^2/2$ term in $V'(r)$ is of no importance as the zero point of a potential is arbitrary. Notice the appearance of a new force term: it is attractive and it scales as $1/r^4$. As a result, it will dominate the h^2/r^3 angular momentum term for sufficiently small r.

We find, therefore, that in general relativity, there is a whole new class of orbits, *capture orbits*, where a particle with non-zero angular momentum h can reach $r = 0$. This is what allows black holes to accrete material.

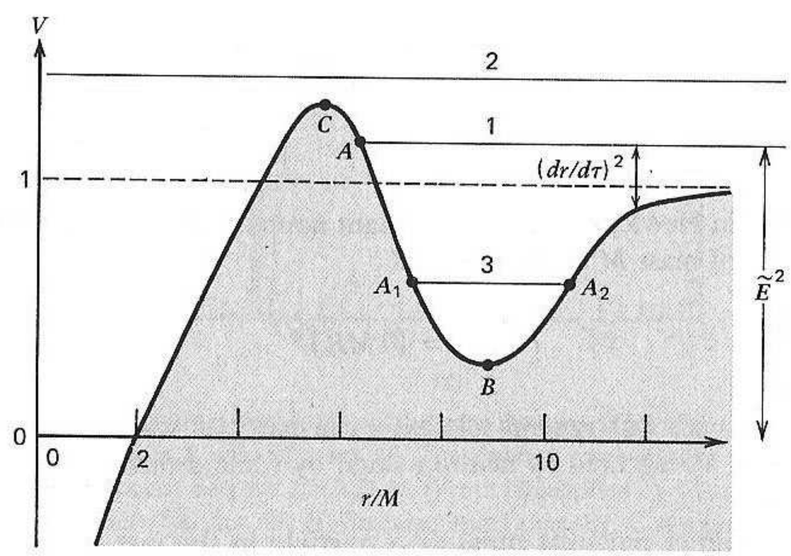


The equivalent one-dimensional potential for inverse square law of force, illustrating bounded motion at negative energies.



Schematic illustration of the nature of the orbits for bounded motion.

Fig. 2.— Left: A particle with energy E_3 is bounded by radii r_1 and r_2. Right: While the orbit will be a closed ellipse for an inverse square force, the orbits for different force laws are not necessarily closed, but are still bounded by r_1 and r_2.



Sketch of the effective potential profile for a particle with *nonzero* rest mass orbiting a Schwarzschild black hole of mass M. The three horizontal lines labeled by different values of $\tilde{E}^2$ correspond to an (1) unbound, (2) capture, and (3) bound orbit, respectively.

1.8. Integration of the Orbit Equation

Now let us look at the solution of eqn(50). Direct integration of this equation to find r as a function of time results in an elliptic integral which is not easy to interpret. It is better to obtain r as a function of θ, as this will give us the shape of the orbit. To make this transformation, we write

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} \qquad \text{but} \qquad \frac{d\theta}{dt} = \dot{\theta} = \frac{h}{r^2} \qquad \text{so} \qquad \frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta} \tag{66}$$

Thus we can write the second derivative of r as

$$\ddot{r} = \frac{d}{dt}\frac{dr}{dt} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) \tag{67}$$

As a result, eqn(50) is now

$$\frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{h^2}{r^3} - \frac{G\ M}{r^2} \tag{68}$$

We now make a further transformation

$$y = \frac{1}{r} \qquad \text{so that} \qquad \frac{dy}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \tag{69}$$

Thus eqn(68) becomes

$$\frac{h}{r^2}\frac{d}{d\theta}\left(-h\frac{dy}{d\theta}\right) = -\frac{h^2}{r^2}\frac{d^2y}{d\theta^2} = \frac{h^2}{r^3} - \frac{G\ M}{r^2} \tag{70}$$

Multiplying through by $-r^2/h^2$, we thus obtain

$$\frac{d^2y}{d\theta^2} + y = \frac{G\ M}{h^2} \tag{71}$$

(We note in passing that the general relativistic potential given in eqn(64) would give rise to the following equation:

$$\frac{d^2y}{d\theta^2} + y = \frac{G\ M}{h^2} + \frac{3GM}{c^2}y^2 \tag{72}$$

The solution to eqn(72) is not a closed orbit, but rather one which looks like a flower – if the last term is small, it is like an ellipse whose axis slowly rotates.)

It is easy to see that a solution of the homogeneous equation $d^2y/d\theta^2 + y = 0$ is $y = \cos(\theta)$. A more general solution is $y = B\cos(\theta - \theta_0)$. Thus we see that if we add the (constant) right hand side of eqn(71), GM/h^2, we obtain the general solution of equation (71):

$$y \; = \; B \; \cos(\theta - \theta_0) \; + \; \frac{GM}{h^2} \tag{73}$$

Recall that $r = 1/y$, so the equation for the orbit is

$$r \; = \; \frac{1}{B \; \cos(\theta - \theta_0) \; + \; \frac{GM}{h^2}} \tag{74}$$

Or

$$r \; = \; \frac{(h^2/GM)}{1 \; + \; \left(\frac{Bh^2}{GM}\right) \cos(\theta - \theta_0)} \tag{75}$$

Now the equation of a *conic section*, which includes the circle, ellipse, parabola and hyperbola, is given by the expression

$$r \; = \; \frac{a(1 - e^2)}{1 \; + \; e \cos(\theta)} \tag{76}$$

So it is clear that our orbit is a conic section, and that we have the relations

$$a(1 - e^2) \; = \; \frac{h^2}{GM} \qquad \text{and} \qquad e \; = \; \frac{Bh^2}{GM} \tag{77}$$

Notice that in eqn(76), as θ varies between 0 and π, r varies from its minimum value of $r_{min} = a(1 - e)$ to its maximum value of $r_{max} = a(1 + e)$. It is clear that θ_0 in eqn(75) is just the phase of the θ variable. We can drop θ_0 if we agree to measure θ from r_{min}.

When r is at r_{min} or r_{max} (call these values r_m), then we are at a *turning point* in the orbit, where $\dot{r} = 0$. At this point, the total velocity is just the transverse velocity: $v = v_\theta = r\dot{\theta}$. So at these points, the kinetic energy per unit mass is

$$T(r_m) \; = \; \frac{1}{2}v^2 \; = \; \frac{1}{2}r_m^2\dot{\theta}^2 \; = \; \frac{h^2}{2r_m^2} \tag{78}$$

Where the last step follows because $\dot{\theta} = h/r^2$ for any r. Now the total energy per unit mass at r_m is

$$\mathcal{E} \; = \; T(r_m) \; + \; V(r_m) \; = \; \frac{h^2}{2r_m^2} \; - \; \frac{GM}{r_m} \tag{79}$$

which we can write as

$$\left(\frac{1}{r_m}\right)^2 \;-\; \frac{2GM}{h^2}\left(\frac{1}{r_m}\right) \;-\; \frac{2\mathcal{E}}{h^2} \;=\; 0 \tag{80}$$

This is a quadratic equation for $1/r_m$. The solution is just

$$\frac{1}{r_m} \;=\; \frac{GM}{h^2} \;\pm\; \left[\frac{G^2M^2}{h^4} + \frac{2\mathcal{E}}{h^2}\right]^{1/2} \tag{81}$$

where the $(+)$ sign corresponds to $r_m = r_{min}$ and the $(-)$ sign to $r_m = r_{max}$. Go back and compare this to eqn(74) for the turning points where $(\theta - \theta_0) = 0$ or π so the cosine is ± 1 :

$$\frac{1}{r_m} \;=\; \frac{GM}{h^2} \;\pm\; B \tag{82}$$

and we see that

$$B \;=\; \left[\frac{G^2M^2}{h^4} + \frac{2\mathcal{E}}{h^2}\right]^{1/2} \tag{83}$$

putting this into eqn(77)

$$e^2 \;=\; \frac{h^4}{G^2M^2}\;B^2 \;=\; 1 \;+\; \frac{2\mathcal{E}h^2}{G^2M^2} \tag{84}$$

and using this for $(1 - e^2)$ in the first part of eqn(77) gives us

$$a\left(-\frac{2\mathcal{E}h^2}{G^2M^2}\right) \;=\; \frac{h^2}{GM} \tag{85}$$

$$\mathcal{E} \;=\; -\;\frac{GM}{2a} \tag{86}$$

Now we also have another expression for the total energy $\mathcal{E}$,

$$\mathcal{E} \;=\; \frac{1}{2}v^2 \;-\; \frac{GM}{r} \tag{87}$$

Here, v is the total velocity, which in general has both radial and transverse components. Combining the above two equations, we obtain the so-called *vis viva* equation:

$$v^2 \;=\; GM\left(\frac{2}{r} \;-\; \frac{1}{a}\right) \tag{88}$$

In the special case of a circular orbit, the semi-major axis a is equal to the (constant) radius r, and we obtain the formula for the velocity in a circular orbit

$$v = \sqrt{\frac{GM}{r}} \tag{89}$$

1.9. Kepler's Third Law

Let us now take eqn(42), which gives the rate, dS/dt, that the radius vector sweeps out area, and integrate it over the whole orbit (for the case 0 $\geq$e< 1, elliptical or circular orbits):

$$\int_{\theta=0}^{\theta=2\pi} dS = \frac{1}{2}h \int_0^P dt = \frac{1}{2}hP \tag{90}$$

where P is the orbital period. But the left hand side is just the total area of the ellipse:

$$\int_{\theta=0}^{\theta=2\pi} dS = \pi ab \tag{91}$$

where b is the semi-minor axis of the ellipse. Since $b = a\sqrt{1-e^2}$,

$$P = \frac{2\pi}{h}ab = \frac{2\pi}{h}a^2(1-e^2)^{1/2} = \frac{2\pi}{h}a^{3/2}a^{1/2}(1-e^2)^{1/2} \tag{92}$$

But from eqn(77), $a^{1/2}(1-e^2)^{1/2} = h/\sqrt{GM}$, so we obtain

$$P = \frac{2\pi}{\sqrt{GM}}\, a^{3/2} \tag{93}$$

which is Kepler's third law.

Perhaps a more usual form is:

$$a^3 = \frac{G(m_1+m_2)}{4\pi^2}\, P^2 \tag{94}$$

Kepler, of course, did not know the physical significance of the constant, but only that $P^2 \propto a^3$.

We should probably note that the gravitational constant G is not known to high accuracy: $G = 6.6739 \times 10^{-8} \pm 0.0001 \times 10^{-8}$ erg cm^2 g^{-2}. Gravity is a *very* weak force and hard to measure in the laboratory. As a result, the mass of the sun is also not that accurately known. On the other hand, the period of the Earth in orbit about the sun (the year) is known accurately, as is the Earth's semi-major axis. Thus from Kepler's 3rd law we can find the product GM_{sun} to high accuracy. As a result, astronomers never use G for accurate work, but rather the *Gaussian gravitational constant*, k , which is defined as $k^2 = GM_{sun}$. Then Kepler's third law for a planet is written

$$a^3 \;=\; \frac{k^2(1+m)}{4\pi^2}\, P^2 \qquad \text{where} \qquad m \;=\; \frac{M_{planet}}{M_{sun}} \tag{95}$$

The Gaussian constant has a value of $k = 0.01720209895$ rad. Yes, the units are radians. In this equation, the unit of length for a is the astronomical unit, the unit of time for P is the mean solar day, and the mass is in solar masses. Now Gauss used $P = 365.256385$ days for the period of the Earth's orbit. Subsequently, astronomers decided to keep the Gaussian value of k, even after they had a better value for the Earth-Sun distance and for the length of the year. So instead of changing k, they adjusted the *astronomical unit* (AU) – which is supposed to be the semi-major axis of the Earth's orbit. As a result, the Earth's semi-major axis is *not* exactly 1 – it is 1.00000106 AU! Just one of those little things astronomers do to keep the physicists out...

1.10. The Shape of Orbits and the Element of Time: Elliptical Orbits

An ellipse can be regarded as the *projection* of a circle. Imagine a circle centered on the ellipse that just touches it at the ends of the major axis. This is the *auxiliary circle.* Imagine tilting this circle about the major axis: its projection is an ellipse. If we draw any line perpendicular to the major axis (see the figure on the next page), then the ratio of the distance between the major axis and the ellipse, RP, to the distance between the axis and the circle, RQ, is is a constant. This is $\mathcal{K}$, the projection factor:

$$\mathcal{K} \;=\; \frac{b}{a} \;=\; \sqrt{1-e^2} \tag{96}$$

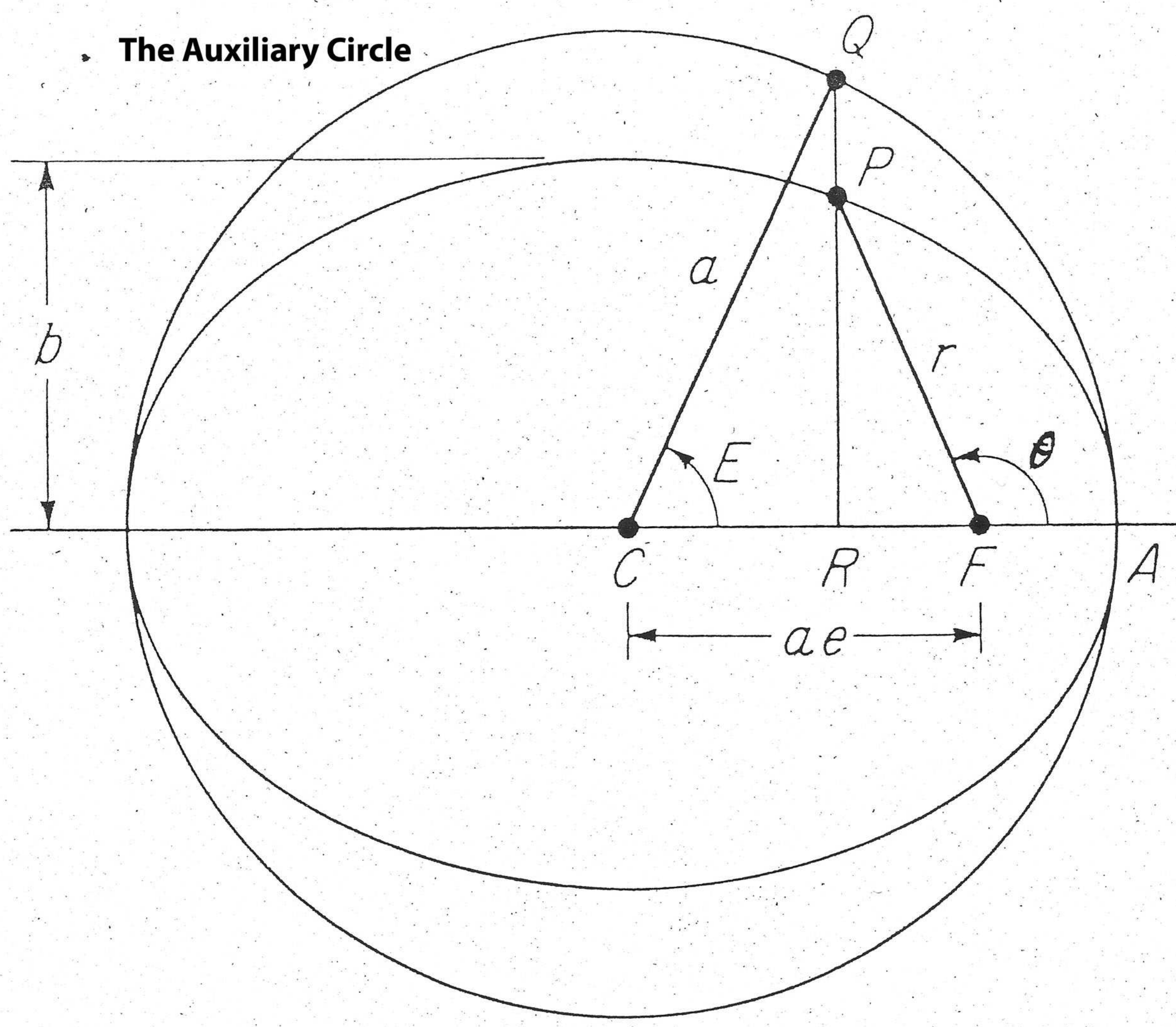
The angle of the line to point Q as measured at the center of the ellipse, is called the *eccentric anomaly*, E. The x-component of this line, CR is just $a\cos E$. Thus the distance FR is just $a\ \cos E\ -\ ae$, negative since we are measuring from the focus F. But we can also express FR in terms of the angle θ , called the *true anomaly.* (The symbol f is often used for θ.) :

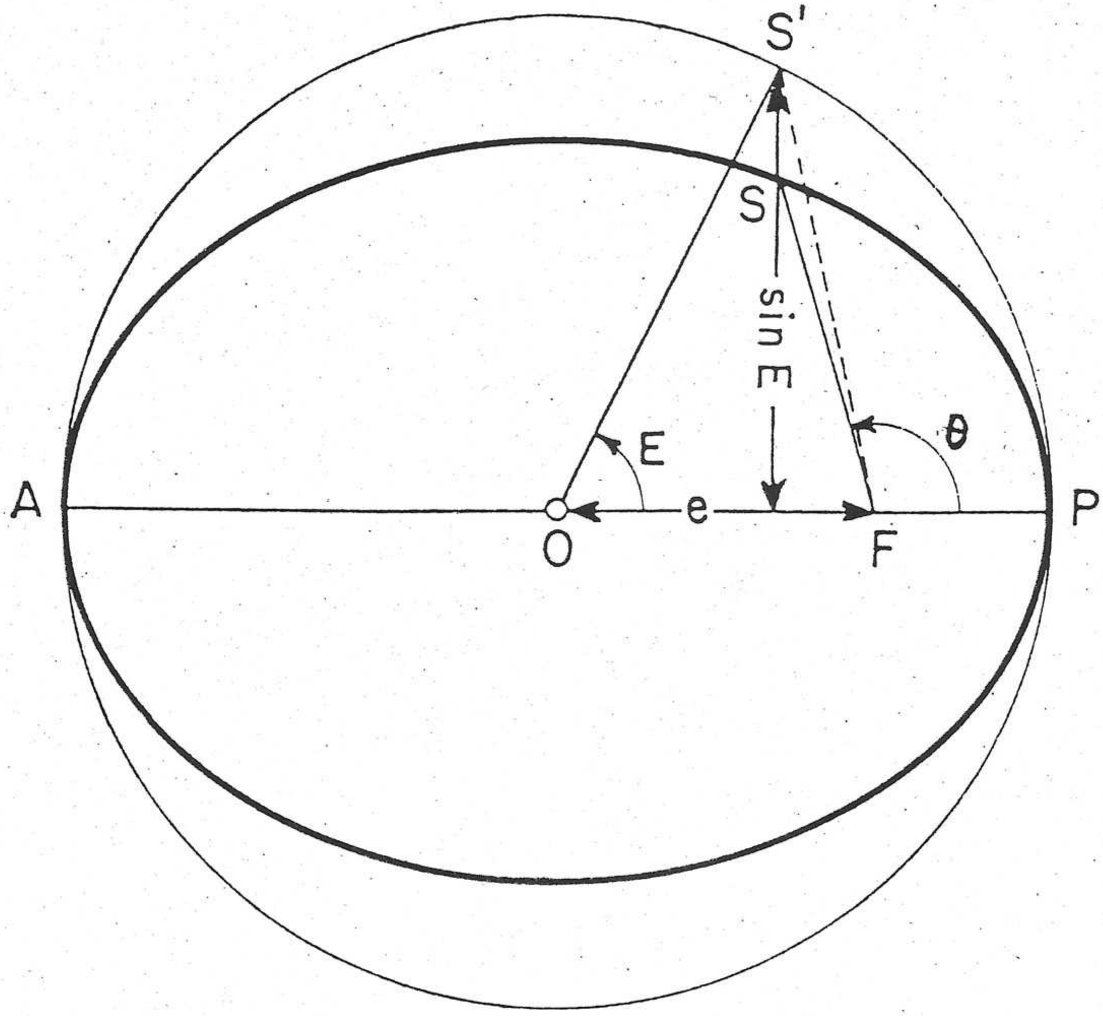
$$r\ \cos\theta \;=\; a\ \cos E \;-\; ae \tag{97}$$

RQ, the y-component of the auxiliary circle, is $a\sin E$. Thus RP, the y-component of the ellipse, is just RQ times the projection factor $\mathcal{K}$. On the other hand, RP is r times $\sin\theta$, so that

$$r\ \sin\theta \;=\; \mathcal{K}\cdot a\ \sin E \;=\; \sqrt{1-e^2}\ a\ \sin E \tag{98}$$

The Auxiliary Circle




S'
S
sin E
E
θ
A
O
e
F
P

If we square these two expressions and add them, after a little manipulation, we obtain the result

$$r \;=\; a\;(1\;-\;e\;\cos E) \tag{99}$$

So r can be expressed in terms of E just as well as in terms of θ using eqn(76). If we equate the two expressions for r, we can derive the relation between the eccentric and the true anomalies:

$$\tan\left(\frac{\theta}{2}\right) \;=\; \sqrt{\frac{1+e}{1-e}}\;\tan\left(\frac{E}{2}\right) \tag{100}$$

The eccentric anomaly is important because it is involved in the simplest expression of the position of the body as a function of time. The key is Kepler's 2nd law, which states that the radius vector sweeps out equal areas of the ellipse in equal time intervals. The second figure introduces a unit orbit and a unit circle, for which $a = 1$. The radius vector FS in the ellipse is just the projection of the vector FS' extending out to the unit circle, shown by the dotted line. The essential point to realize is this: as the vector FS sweeps out area, FS' sweeps out area on the circle at a rate proportional to that on the ellipse, the ratio always being (ellipse)/(circle)= $\mathcal{K}$, the projection factor. Now the area of the unit circle is π, so the area in the sector $FS'P$ must amount to $\pi(t-t_0)/P$, where t_0 is the time of perihelion passage (point P).

Now the area of the sector of the circle $FS'P$ is just the area of the sector $S'OP$ minus the area of the triangle $OS'F$. The area of $S'OP$ is $E/2$ (where E is in radians). The area of the triangle is (1/2)(base)(height) = (1/2)(e)($\sin E$). Thus the swept area $FS'P =$ $(E/2) - (e\sin E/2)$. Equating the two expressions for the area $FS'P$ then leads to

$$\pi\frac{t-t_0}{P} \;=\; \frac{E}{2} \;-\; \frac{e\;\sin E}{2} \tag{101}$$

Let us define the *mean anomaly*, M (don't confuse it with the total mass M!):

$$M \;=\; \frac{2\pi}{P}(t-t_0) \tag{102}$$

As the time goes from 0 to one orbital period, M increases uniformly from 0 to 2π. Thus we finally arrive at **Kepler's equation**:

$$M \;=\; E \;-\; e\;\sin E \tag{103}$$

We thus have all we need to find where the planet will be at time t :

- From t, t_0, and P, use eqn(102) to find M.
- From M and e, use eqn(103) to find E.
- From E and e, use eqn(100) to find θ.
- From E, e and a, use eqn(99) to find r. Or use θ and eqn(76).

To carry out this procedure, we would like to invert Kepler's equation (eqn 103) to give E as an explicit function of M. Unfortunately, this is a transcendental equation which cannot be inverted to give a closed expression for E. Over the years, literally hundreds of methods have been proposed for the solution of Kepler's equation.

If e is small, then a series expansions may be constructed, e.g.,

$$E \simeq M + e \sin M + \frac{e^2}{2}\sin 2M + \frac{e^3}{8}(3\sin 3M - \sin M) + \cdots \tag{104}$$

And such a series can even be put into eqn(100) to give θ directly as a series expansion:

$$\theta \simeq M + 2e \sin M + \frac{5}{4}e^2 \sin 2M + \frac{1}{12}e^3(13\sin 3M - 3\sin M) + \cdots \tag{105}$$

Before computers were available to solve such problems, volumes of hairy math were devoted to this and related problems. One important representation of the problem is

$$E = M + 2\sum_{n=1}^{\infty}\frac{1}{n} J_n(ne)\sin(nM) \tag{106}$$

where J_n is the *Bessel function* of the first kind and of order n. This is one of the first places that Bessel functions appeared. If terms of order e^6 or higher are neglected, the first few J_n in eqn(106) are

$$\begin{aligned}
J_1(e) &= \frac{1}{2}e\left(1 - \frac{1}{8}e^2 + \frac{1}{192}e^4\right)\\
J_2(2e) &= \frac{1}{2}e^2\left(1 - \frac{1}{3}e^2\right)\\
J_3(3e) &= \frac{9}{16}e^3\left(1 - \frac{9}{16}e^2\right)
\end{aligned}$$

The problem with these expansions is that if e is close to unity (e.g. $e = 0.967$ for Halley's comet) the expansions converge very slowly – eqn(105) may even diverge.

A better approach is to find the solution by iteration. We could simply rewrite Kepler's equation as

$$E \;=\; M \;+\; e\;\sin E \tag{107}$$

and setting $E = M$ on the right hand side, evaluate a new E, and then repeat this procedure until the equation returns the same E. But convergence may be rather slow.

The Newton-Raphson method is an iterative procedure which converges more rapidly (the significant figures will eventually double with each iteration). The downside is that sometimes it will not converge at all. It seems good for this problem. Suppose we want to find the solution of the equation $f(x) = 0$, where $f(x)$ is a function we can't invert algebraically. The Newton-Raphson method is to guess a value x_0 for the solution and then iterate according to the formula

$$x_{i+1} \;=\; x_i \;-\; \frac{f(x_i)}{f'(x_i)} \qquad \text{where} \qquad f'(x) = \frac{df}{dx} \;\; . \tag{108}$$

The Newton-Raphson formula applied to Kepler's equation is just

$$E_{n+1} \;=\; E_n \;+\; \frac{M - E_n + e\sin E_n}{1 - e\cos E_n} \qquad \text{for} \qquad n = 0, 1, 2, \cdots \tag{109}$$

We pick a first guess, say $E_0 = M$, and iterate to convergence.