1. The Restricted Three-Body Problem

The *restricted three-body problem* refers to the motion of a third body of very small mass $m << M_1, M_2$ moving in the gravitational field of two bodies of masses M_1 and M_2 which are in circular orbits about their common center of mass. We will analyze this problem by considering a coordinate system which is rotating with M_1 and M_2 so that the center of mass is the origin and the two large masses remain on the x-axis.

The position of the small mass m is given by the vector $\vec{r}$. We recall that the acceleration $\ddot{\vec{r}}$ in the r, θ coordinate system is given by

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} \tag{1}$$

Now our coordinate system is rotating with an angular velocity $\omega = \dot{\theta}$, where ω is constant. If the particle were at rest in this rotating system, then $\dot{r} = 0$. Since $\dot{\theta}$ is constant, $\ddot{\theta} = 0$, and the acceleration is then

$$\ddot{\vec{r}} = -r\,\dot{\theta}^2\,\hat{r} = -r\omega^2\,\hat{r} \tag{2}$$

Since a particle at rest in the rotating coordinates has this acceleration, this is, by $\vec{F} = m\vec{a}$, the force needed to keep it at rest, the *centripetal force*. Conversely, objects in this rotating frame will seem to experience a *centrifugal force* given by $r\omega^2\,\hat{r}$ pushing them away from the origin. This force can be derived from a potential $\Phi_c = -(1/2)\omega^2 r^2$ such that $-\nabla\Phi_c = r\omega^2\,\hat{r}$. So now we see how to write the potential (per unit mass) of the particle in the rotating coordinate system: it is just the gravitational potential of the two masses M_1 and M_2 at the position of m plus the Φ_c term:

$$\Phi = -\frac{GM_1}{s_1} - \frac{GM_2}{s_2} - \frac{1}{2}\omega^2 r^2 \tag{3}$$

Now the angular velocity (which is in radians/sec) is just the number of radians in a circle divided by the period of the rotating masses M_1 and M_2. Thus we have that

$$\omega^2 = \left(\frac{2\pi}{P}\right)^2 = \frac{G(M_1 + M_2)}{a^3} \tag{4}$$

where the last term just follows from Kepler's 3rd law. We can thus write the potential of the particle as

$$\Phi = -G\left\{\frac{M_1}{s_1} + \frac{M_2}{s_2} + \frac{1}{2}\frac{(M_1 + M_2)}{a^3}\,r^2\right\} \tag{5}$$

Now we let M denote the total mass $M = M_1 + M_2$, and we define μ as

$$\mu = \frac{M_2}{M_1 + M_2} \quad \text{where} \quad M_2 \le M_1\,, \tag{6}$$

so that $\mu \le 1/2$. Thus $M_2 = \mu M$ and $M_1 = (1 - \mu)M$ and we thus can write

$$-\frac{\Phi}{GM} = \frac{1-\mu}{s_1} + \frac{\mu}{s_2} + \frac{1}{2}\frac{r^2}{a^3}\,. \tag{7}$$

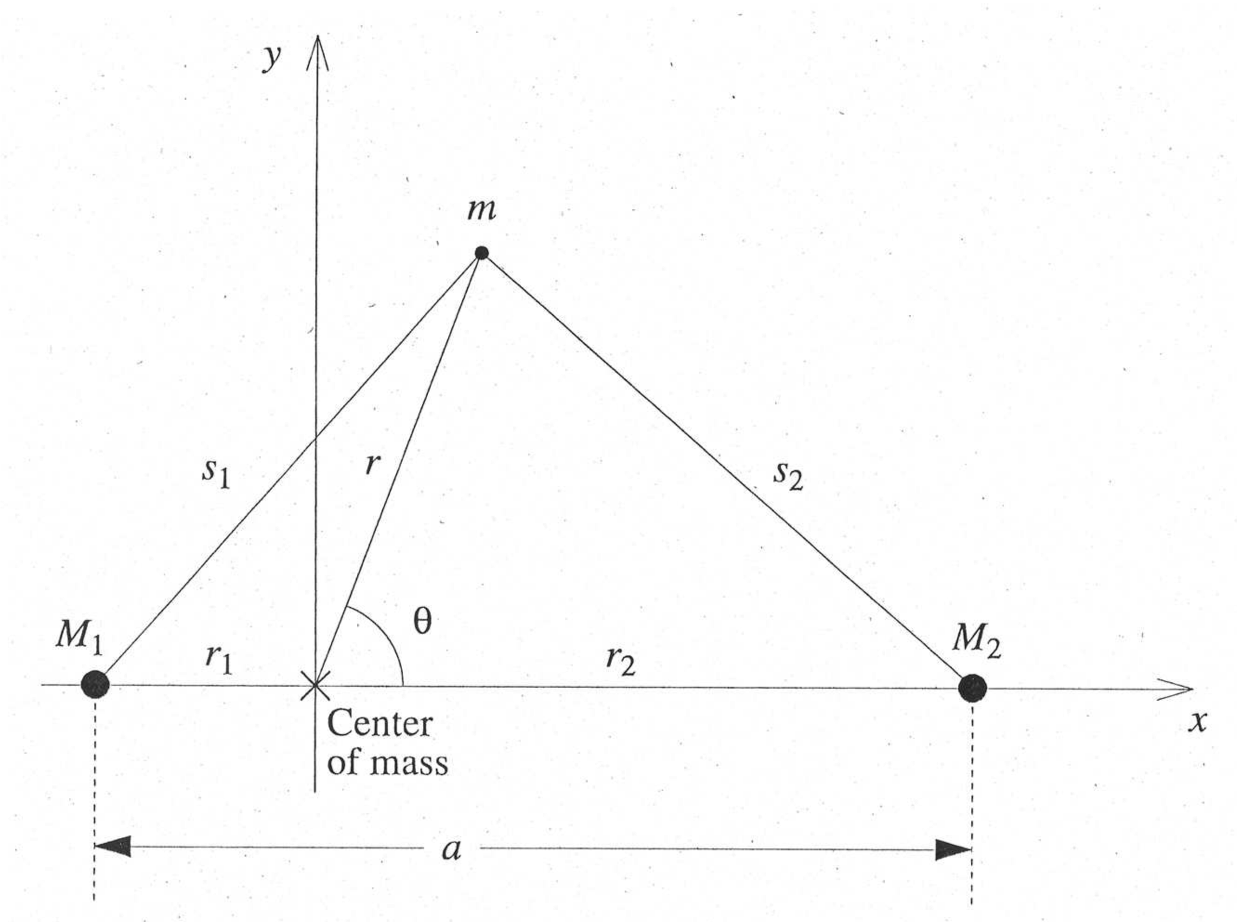

Fig. 1.— The corotating coordinates for the three-body problem.

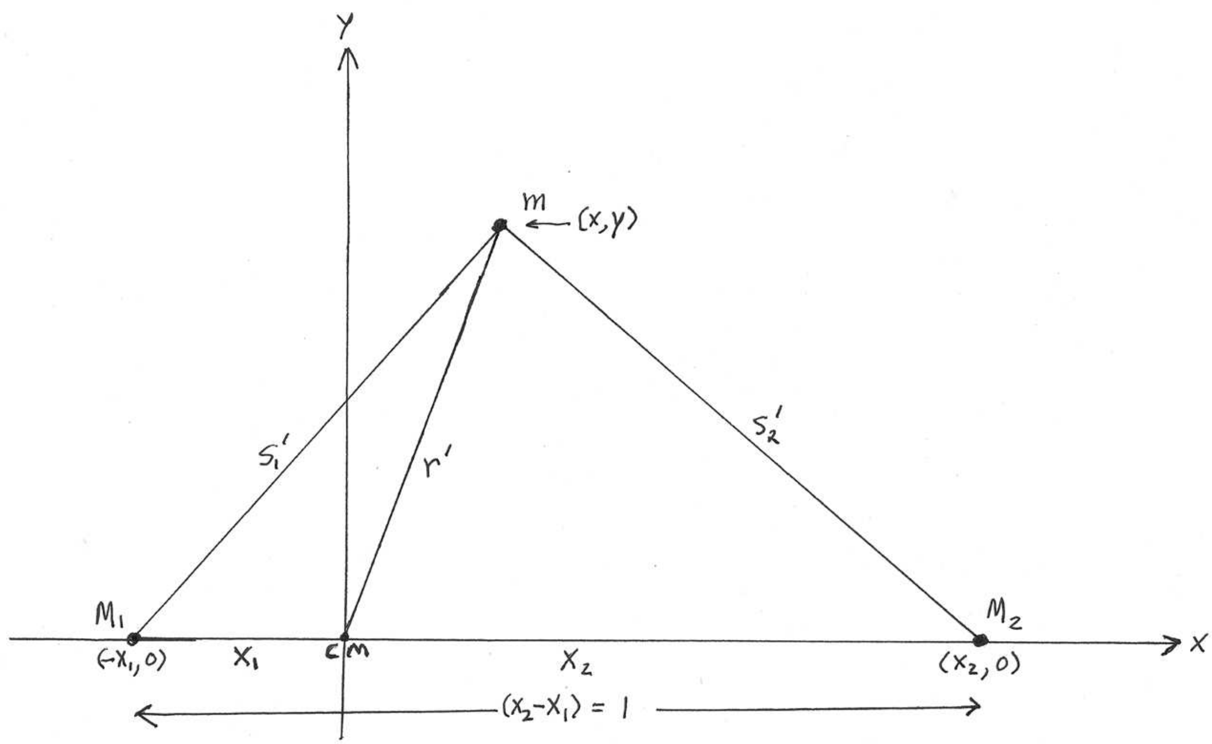

Fig. 2.— The coordinates normalized so that $a = ((-x_1) + x_2) = 1$.

The next step is to redefine the length so that a, the distance between M_1 and M_2, becomes unity. We thus introduce $s_1' = s_1/a$, $s_2' = s_2/a$ and $r' = r/a$, and write the equation as

$$-\frac{\Phi}{GM} = \frac{1-\mu}{as_1'} + \frac{\mu}{as_2'} + \frac{1}{2}\frac{(ar')^2}{a^3} . \tag{8}$$

and multiplying through by a we find a scaled potential V':

$$-V' = -\frac{a}{GM}\Phi = \frac{1-\mu}{s_1'} + \frac{\mu}{s_2'} + \frac{1}{2}(r')^2 . \tag{9}$$

Now M_1 and M_2 are on the x-axis and the origin is the center of mass. M_1 is left of the origin, so it's coordinate x_1 is negative. The distance between M_1 and M_2 is thus $(-x_1) + x_2 = (x_2 - x_1) = 1$. The coordinates of our particle m are (x,y) and the distance of m from the origin is $(r')^2 = x^2 + y^2$. We also see that by drawing a line from (x,y) perpendicular to the x-axis that $(s_1')^2 = (x - x_1)^2 + y^2$ and $(s_2')^2 = (x - x_2)^2 + y^2$. Also, from $M_2(x_2) = M_1(-x_1)$ we have $\mu x_2 = -(1-\mu)x_1$, which, combined with $(x_2 - x_1) = 1$, yields $x_1 = -\mu$ and $x_2 = (1-\mu)$. Substituting these values into the foregoing we obtain

$$-V' = \frac{1-\mu}{([x-x_1]^2 + y^2)^{1/2}} + \frac{\mu}{([x-x_2]^2 + y^2)^{1/2}} + \frac{1}{2}(x^2 + y^2) . \tag{10}$$

Now the force on m will be proportional to the gradient of the potential $\nabla V'$. For the particle to be at rest, the forces must vanish: $F_x = 0$ and $F_y = 0$. This will happen if

$$\frac{\partial V'}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial V'}{\partial y} = 0 . \tag{11}$$

Let us first look at the y derivative:

$$\frac{\partial(-V')}{\partial y} = y + (1-\mu)\frac{\partial}{\partial y}\left([x-x_1]^2 + y^2\right)^{-1/2} + \mu\frac{\partial}{\partial y}\left([x-x_2]^2 + y^2\right)^{-1/2} \tag{12}$$

which results in

$$\frac{\partial(-V')}{\partial y} = y - \frac{(1-\mu)\, y}{([x-x_1]^2 + y^2)^{3/2}} - \frac{\mu\, y}{([x-x_2]^2 + y^2)^{3/2}} = 0 \tag{13}$$

We see that one solution (though not the only one) is simply $y = 0$. This will give the equilibrium points on the x-axis. We thus proceed by first setting $y = 0$.

On the x-axis, the condition that $\partial(-V')/\partial x = 0$ then leads to the equation

$$x - \frac{(1-\mu)(x-x_1)}{|x-x_1|^3} - \frac{\mu(x-x_2)}{|x-x_2|^3} = 0 \tag{14}$$

where the absolute values are necessary because the denominators – which are the cubes of the distances s_1' and s_2' – must be positive.

Consider the solution for $x > x_2$, i.e., the L_2 point. Then the arguments of the absolute values are positive and we have

$$x \;-\; \frac{(1-\mu)}{(x-x_1)^2} \;-\; \frac{\mu}{(x-x_2)^2} \;=\; 0 \tag{15}$$

But from $(1-\mu)x_1 = -\mu x_2$, we have $x_1 = -\mu$ and $x_2 = 1-\mu$. If we define $\nu = 1-\mu$, then eqn (15) becomes

$$x \;-\; \frac{\nu}{(x+\mu)^2} \;-\; \frac{\mu}{(x-\nu)^2} \;=\; 0 \tag{16}$$

While this looks simple, if we multiply by the product of the denominators, expand and collect coefficients, we get a rather messy 5th order polynomial:

$$\begin{aligned} x^5 \;-2(\nu-\mu)x^4 \;+\; (\nu^2 - 4\nu\mu + \mu^2)x^3 \;+\; (2\nu\mu(\nu-\mu) - 1)x^2 \;+ \\ (\nu^2\mu^2 + 2(\nu-\mu))x \;-\; (\nu^3 + \mu^3) \;=\; 0 \end{aligned} \tag{17}$$

Thus, for example, in the case of equal masses, $\mu = \nu = 1/2$, we have

$$x^5 \;-\; \frac{1}{2}x^3 \;-\; x^2 \;+\; \frac{1}{16}x \;-\; \frac{1}{4} \;=\; 0 \tag{18}$$

The solution of this equation is $x = L_2 = 1.19840614$. In this case, because of the symmetry, $L_3 = -L_2$, and $L_1 = 0$. To get the quintic equation in terms of μ alone, we substitute back $\nu = 1-\mu$ and obtain

$$\begin{aligned} x^5 \;+\; (4\mu-2)x^4 \;+\; (6\mu^2 - 6\mu + 1)x^3 \;+\; (4\mu^3 - 6\mu^2 + 2\mu - 1)x^2 \;+ \\ (\mu^4 - 2\mu^3 + \mu^2 - 4\mu + 2)x \;-\; (3\mu^2 - 3\mu + 1) \;=\; 0 \end{aligned} \tag{19}$$

As a check, we can look at the limit as $\mu \to 0$. We see that eqn (19) then becomes

$$x^5 \;-\; 2x^4 \;+\; x^3 \;-\; x^2 \;+\; 2x \;-\; 1 \;=\; 0 \tag{20}$$

which has the expected solution $x = 1$.

Probably the simplest expression results if we change the variable from x to $\rho = x - x_2 = x - \nu = x - (1-\mu)$. If we do this, eqn (16) becomes

$$(\rho + 1 - \mu) \;-\; \frac{1-\mu}{(1+\rho)^2} \;-\; \frac{\mu}{\rho^2} \;=\; 0 \tag{21}$$

which expands to

$$\rho^5 \;+\; (3-\mu)\rho^4 \;+\; (3-2\mu)\rho^3 \;-\; \mu\rho^2 \;-\; 2\mu\rho \;-\; \mu \;=\; 0 \tag{22}$$

We solve this equation for the single real root ρ and then recover x from $x = \rho+1-\mu$. To find the Lagrangian point L_1, the point between the two masses, we define ρ as $\rho = (1-\mu) - x$. Then the quintic becomes

$$\rho^5 \;-\; (3-\mu)\rho^4 \;+\; (3-2\mu)\rho^3 \;-\; \mu\rho^2 \;+\; 2\mu\rho \;-\; \mu \;=\; 0 \tag{23}$$

Finally, to get L_3, the point on the opposite side of M_1 from M_2, we must take $\rho = x + (1+\mu)$. This leads to

$$\rho^5 \;-\; (7+\mu)\rho^4 \;+\; (19+6\mu)\rho^3 \;-\; (24+13\mu)\rho^2 \;+\; (12+14\mu)\rho \;-\; 7\mu \;=\; 0 \tag{24}$$

While equations (22), (23) and (24) must be solved numerically in general, there are useful approximations for the case where one mass is much smaller than the other, i.e., $\mu << 1$. In this case, to the lowest order in μ, the locations of the Lagrangian points are found to be

$$x(L_1) = 1 \;-\; [\mu/3]^{1/3} \quad , \quad x(L_2) = 1 \;+\; [\mu/3]^{1/3} \quad , \quad x(L_3) = -\left[1 \;+\; \frac{5}{12}\mu\right] \tag{25}$$

Since $\mu << 1$, the term $[\mu/3]^{1/3}$ will be much larger than $(5/12)\mu$. For example, if $\mu = 0.003$, $[\mu/3]^{1/3} = 0.1$, while $(5/12)\mu = 0.00125$. Thus L_1 and L_3 will be displaced interior and exterior to m_2 by a substantial amount, while L_3 will be opposite m_2 but only very slightly beyond the orbit of m_2 about m_1.

The approximate sphere around M_2 which touches L_1 and L_2 is called the *Hill sphere*. Within this region, the force is directed toward M_2, so that a body may orbit M_2 without wandering into an orbit about M_1. For example, the Moon is well within the Hill sphere of Earth-Sun system. Using the approximations of eqn (25), we find that the radius of the Hill sphere about the smaller mass is

$$R_{Hill} \;=\; [\mu/3]^{1/3}\, a \tag{26}$$

Consider the case where M_1 is the Sun and M_2 the Earth. Then

$$\mu \;=\; \frac{M_\oplus}{M_\odot + M_\oplus} \;=\; \frac{1}{(M_\odot/M_\oplus)+1} \;=\; \frac{1}{332946+1} \;=\; 3.00348 \times 10^{-6} \tag{27}$$

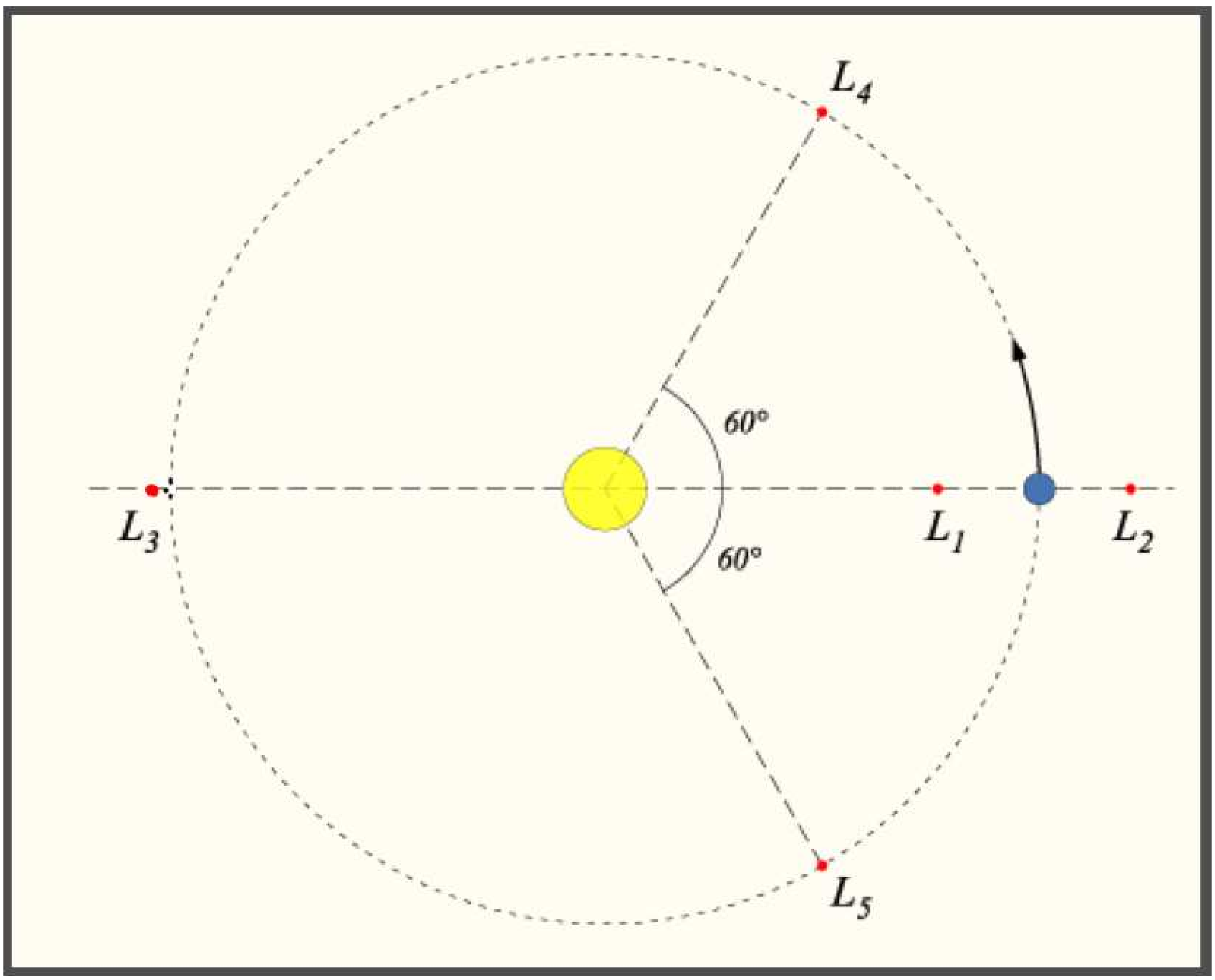

Fig. 3.— The five Lagrange points.

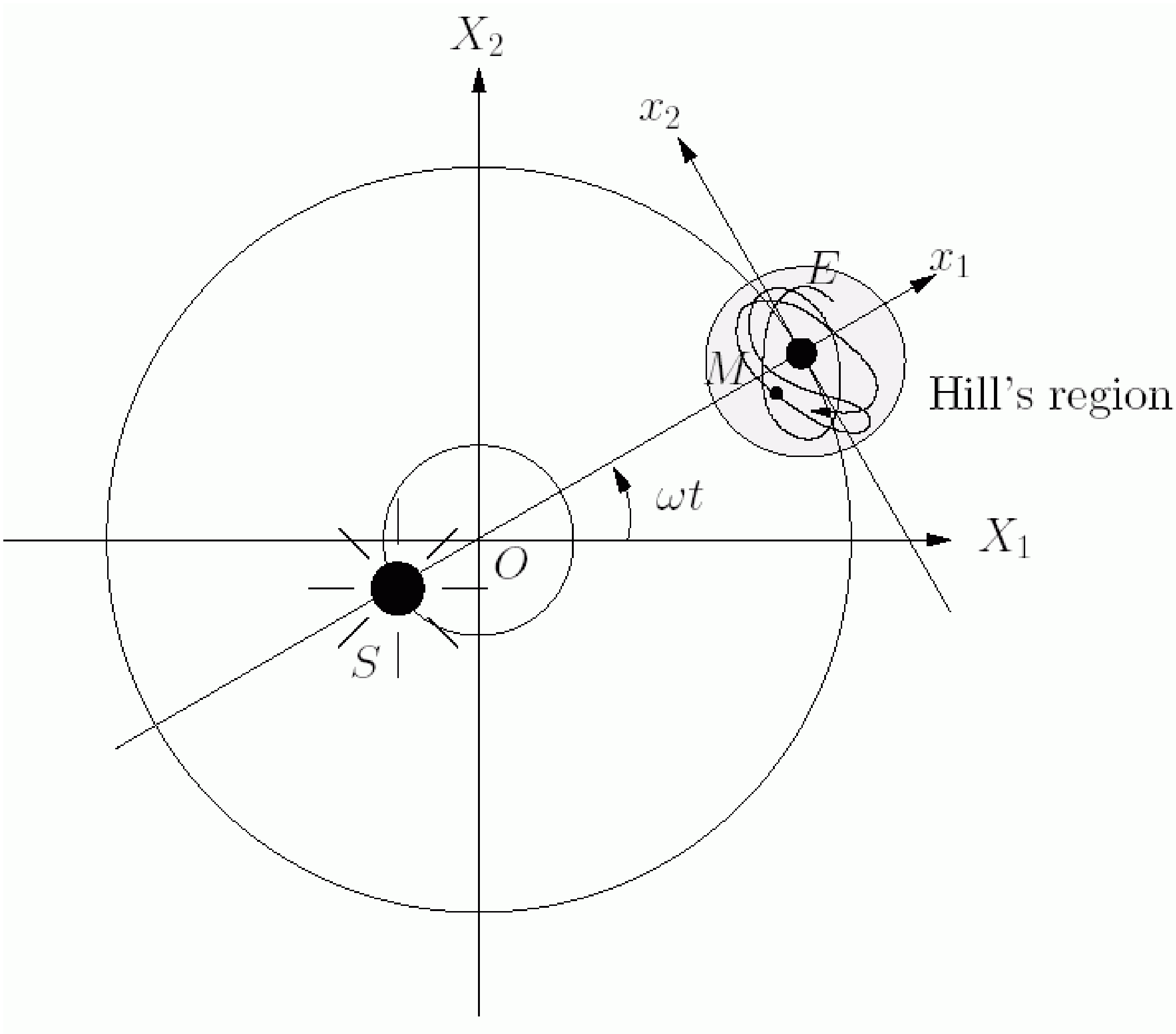

Fig. 4.— The Hill sphere about the mass M_2.

We can then solve for the positions: $L_1 = 0.990027$ and $L_2 = 1.01003$, in units of the Earth-Sun seperation, which in this case is AU. The distances of L_1 and L_2 *from the Earth* are $L_1 = -1.4916 \times 10^6$ km and $L_2 = 1.5015 \times 10^6$ km. For comparison, the Earth-Moon distance is 0.3844×10^6 km, so the Moon is well within the Earth's Hill sphere.

But what about the possibility that y is not zero? Then we proceed by dividing the y-equation (13) by y:

$$1 \;-\; \frac{(1-\mu)}{([x-x_1]^2+y^2)^{3/2}} \;-\; \frac{\mu}{([x-x_2]^2+y^2)^{3/2}} \;=\; 0 \tag{28}$$

Now the x equation, $\partial(-V')/\partial x = 0$, when we include y is

$$x \;-\; \frac{(1-\mu)(x-x_1)}{([x-x_1]^2+y^2)^{3/2}} \;-\; \frac{\mu(x-x_2)}{([x-x_2]^2+y^2)^{3/2}} \;=\; 0 \tag{29}$$

Now multiply eqn (28) by $(x-x_2)$ to obtain

$$(x-x_2) \;-\; \frac{(1-\mu)(x-x_2)}{([x-x_1]^2+y^2)^{3/2}} \;-\; \frac{\mu(x-x_2)}{([x-x_2]^2+y^2)^{3/2}} \;=\; 0 \tag{30}$$

and subtract from eqn (29) to obtain

$$x_2 \;-\; \frac{(1-\mu)(x_2-x_1)}{([x-x_1]^2+y^2)^{3/2}} \;=\; 0 \tag{31}$$

In the same way, if we multiply by $(x-x_1)$ and subtract we obtain

$$x_1 \;-\; \frac{\mu(x_1-x_2)}{([x-x_2]^2+y^2)^{3/2}} \;=\; 0 \tag{32}$$

But recall that $(x_2 - x_1) = 1$, $x_2 = (1-\mu)$, and $x_1 = -\mu$. It follows from eqn (31) that

$$(1-\mu) \;-\; \frac{(1-\mu)}{([x-x_1]^2+y^2)^{3/2}} \;=\; 0 \tag{33}$$

from which we find that

$$\left([x-x_1]^2+y^2\right)^{1/2} \;=\; s_1' \;=\; 1 \tag{34}$$

and, just the same, from (32) we have

$$-\mu \;-\; \frac{\mu(-1)}{([x-x_2]^2+y^2)^{3/2}} \;=\; 0 \tag{35}$$

which leads immediately to

$$\left([x-x_2]^2+y^2\right)^{1/2} \;=\; s_2' \;=\; 1 \tag{36}$$

Thus the equilibrium points are where $s_1' = s_2' = ((-x1) + x2) = 1$: they form an equilateral triangle with M_1 and M_2. Remarkable!